

Algebras with homogeneous module category are tame

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Abstract

The celebrated Drozd's theorem asserts that a finite-dimensional basic algebra Λ over an algebraically closed field k is either tame or wild, whereas the Crawley-Boevey's theorem states that given a tame algebra Λ and a dimension d , all but finitely many isomorphism classes of indecomposable Λ -modules of dimension d are isomorphic to their Auslander-Reiten translations and hence belong to homogeneous tubes. In this paper, we prove the inverse of Crawley-Boevey's theorem, which gives an internal description of tameness in terms of Auslander-Reiten quivers.

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References

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Introduction

Throughout the paper, we always assume that k is an algebraically closed field, and that all rings or algebras contain identities. We write our maps either on the left or on the right, but always compose them as if they were written on the right.

We start with the following important definition of “tame” and “wild”:

Definition 1 [D1, CB1, DS] A finite-dimensional k -algebra Λ is of tame representation type, if for any positive integer d , there are a finite number of localizations $R_i = k[x, \phi_i(x)^{-1}]$ of $k[x]$ and Λ - R_i -bimodules T_i which are free as right R_i -modules, such that all but finitely many iso-classes of indecomposable Λ -modules of dimension at most d are isomorphic to

$$T_i \otimes_{R_i} R_i/(x - \lambda)^m,$$

for some $\lambda \in k$ with $\phi_i(\lambda) \neq 0$, and some positive integer m .

A finite-dimensional k -algebra Λ is of wild representation type if there is a finitely generated Λ - $k\langle x, y \rangle$ -bimodule T , which is free as a right $k\langle x, y \rangle$ -module, such that the functor

$$T \otimes_{k\langle x, y \rangle} - : k\langle x, y \rangle\text{-mod} \rightarrow \Lambda\text{-mod}$$

preserves indecomposability and isomorphism classes.

Several authors worked on equivalent definitions of “tame” and “wild”, for example in terms of generic modules [CB3].

In 1977 Drozd [D1] showed that a finite-dimensional algebra over an algebraically closed field is either of tame representation type or of wild representation type. This result is known as Drozd’s Tame-Wild Theorem, and has been one of the most fundamental results in the representation theory of finite dimensional algebras. On the other hand, however, the proof of Drozd’s Theorem is highly indirect. Indeed, the argument relies on the notion of a boc (the abbreviation for “bimodule of coalgebra structure”), introduced first by Rojter in [Ro]. In 1988, Crawley-Boevey [CB1] formalized the theory of bocses and showed that for a tame algebra Λ , and for each dimension d , all but finitely many isomorphism classes of indecomposable Λ -modules of dimension d are isomorphic to their Auslander-Reiten translations and hence belong to homogeneous tubes.

After the work [CB1], many authors tried to prove the converse of Crawley-Boevey’s theorem, aiming to find infinitely many non-isomorphic indecomposable representations $\{M_i \mid i \in I\}$ of the same dimension in the representation category of a layered boc, such that $M_i \not\cong DTr(M_i)$. Somewhat surprisingly, in 2000 the authors [BCLZ] constructed a strongly homogeneous wild layered boc \mathfrak{B} for which each representation is homogeneous (i.e., $DTr(M) \simeq M$), and showed that the converse of the Crawley-Boevey’s theorem does not hold true for general layered bocses. Later on, C.M. Ringel proposed a concept of controlled wild, Y. Han described some classes of controlled wild algebras [H], and H. Nagasy proved that a τ -wild algebra is wild [N]. However, the converse of the Crawley-Boevey’s theorem remains open in the case of finite dimensional k -algebras. Our main result in this paper, Theorem 3 below, gives a full answer to this problem. To state our result, we need the following definition:

Definition 2 [BCLZ, 2.1 Definitions] Let Λ be a finite-dimensional algebra over an algebraically closed field. An indecomposable Λ -module M is called *homogeneous*, if $DTr(M) \cong M$. The category $\text{mod-}\Lambda$ is said to be *homogeneous*, if for each dimension d all but finitely many isomorphism classes of indecomposable Λ -modules of dimension d are homogeneous.

We will prove the following main theorem throughout this whole article.

Theorem 3 Let Λ be a finite-dimensional algebra over an algebraically closed field. Then Λ is of tame representation type if and only if $\text{mod-}\Lambda$ is homogeneous. \square

The necessity of Theorem 3 was previously proved by Crawley-Boevey [CB1]. We only need to prove the sufficiency part. The proof is divided into five sections as shown in the contents. Our proof relies on the notions of matrix bimodule problems, their associated bocses, and reduction techniques. Since the matrix bimodule problems associated to finite-dimensional algebras are bipartite, the key of our argument is to find a full subcategory of representation category of a bipartite matrix bimodule problem which admits infinitely many isomorphism classes of non-homogeneous representations of dimension d .

1 Matrix bimodule problems

In this section, a notion of matrix bimodule problems over a minimal algebra is introduced, which is a generalization of bimodule problems over a field k defined by [CB2] in terms of matrix. Then the associated bi-comodule problems and bocses of matrix bimodule problems are discussed. Finally, a nice connection between a matrix bimodule problem and its associated bocs is build via the formal products of two structures.

1.1 Definition of matrix bimodule problems

The purpose of this subsection is two folds: 1) construct a k -algebra Δ based on a minimal algebra R ; 2) define matrix bimodule problems over Δ . The concepts and the results are proposed by S. Liu.

Let $\mathcal{T} = \mathcal{T}_0 \dot{\cup} \mathcal{T}_1$ be a vertex set, where the subset \mathcal{T}_0 consists of trivial vertices, such that $\forall X \in \mathcal{T}_0$, there is a k -algebra $R_X \simeq k$ with the identity 1_X ; and \mathcal{T}_1 consists of non-trivial vertices, such that $\forall X \in \mathcal{T}_1$, there is an algebra $R_X \simeq k[x, \phi_X(x)^{-1}]$ with the identity 1_X , the finite localization of the polynomial ring $k[x]$ given by a non-zero polynomial $\phi_X(x) \in k[x]$, and x is said to be the *parameter associated to* $X \in \mathcal{T}_1$. Now we call the k -algebra $R = \prod_{X \in \mathcal{T}} R_X$ a *minimal algebra* over \mathcal{T} with a set of orthogonal primitive idempotents $\{1_X \mid X \in \mathcal{T}\}$.

We define a tensor product of $p \geq 1$ copies of R over k as follows:

$$R^{\otimes p} = R \otimes_k \cdots \otimes_k R = \sum_{(X_1, \dots, X_p) \in \mathcal{T} \times \cdots \times \mathcal{T}} R_{X_1} \otimes_k \cdots \otimes_k R_{X_p}.$$

There exists a natural left and right R -module structure on $R^{\otimes p}$:

$$\begin{aligned} s \otimes_R \alpha &= (s \otimes_R r_1) \otimes_k r_2 \otimes_k \cdots \otimes_k r_p; \\ \alpha \otimes_R s &= r_1 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k (r_p \otimes_R s), \end{aligned} \quad (1.1-1)$$

for any $\alpha = r_1 \otimes_k r_2 \otimes_k \cdots \otimes_k r_{p-1} \otimes_k r_p \in R^{\otimes p}$, $s \in R$. If $r_i \in R_{X_i}$, $s \in R_Y$, then $s \otimes_R \alpha = 0$ for $Y \neq X_1$ and $\alpha \otimes_R s = 0$ for $Y \neq X_p$. Thus $R^{\otimes p}$ can be viewed as an R - R -bimodule, or simply an $R^{\otimes 2}$ -module, with the module action for any $r, s \in R$:

$$(r \otimes_k s) \otimes_{R^{\otimes 2}} \alpha = r \otimes_R \alpha \otimes_R s = \alpha \otimes_{R^{\otimes 2}} (r \otimes_k s). \quad (1.1-2)$$

Note that $(\alpha \otimes_{R^{\otimes 2}} (r \otimes_k s)) \otimes_R s' = \alpha \otimes_{R^{\otimes 2}} (r \otimes_k s s'), \forall s' \in R$, and $r' \otimes_R (\alpha \otimes_{R^{\otimes 2}} (r \otimes_k s)) = \alpha \otimes_{R^{\otimes 2}} (r' r \otimes_k s), \forall r' \in R$. The direct sum of $R^{\otimes p}$ for $p = 1, 2, \dots$, is still an $R^{\otimes 2}$ -module:

$$\Delta = \bigoplus_{p=1}^{\infty} R^{\otimes p}, \quad \text{let } \bar{\Delta} = \bigoplus_{p=2}^{\infty} R^{\otimes p}, \quad \Delta = R \oplus \bar{\Delta}. \quad (1.1-3)$$

We define a multiplication on $R^{\otimes 2}$ -module Δ , given by $\Delta \times \Delta \rightarrow \Delta \otimes_R \Delta \subseteq \Delta$:

$$\Delta^{\otimes p} \otimes_R \Delta^{\otimes q} \subseteq \Delta^{\otimes (p+q-1)}, \quad \alpha \otimes_R \beta = r_1 \otimes_k \cdots \otimes_k (r_p s_1) \otimes_k s_2 \cdots \otimes_k s_q, \quad (1.1-4)$$

where $\beta = s_1 \otimes_k \cdots \otimes_k s_q$. And if $r_i \in R_{X_i}, s_j \in R_{Y_j}$, $\alpha \otimes_R \beta = 0$ for $X_p \neq Y_1$. Thus we obtain an associative non-commutative k -algebra $(\Delta, \otimes_R, 1_R)$ with the set of orthogonal primitive idempotents $\{1_X \mid X \in \mathcal{T}\}$. Moreover, $\Delta \otimes_R \Delta$ can be viewed as an $R^{\otimes 3}$ -module: for any $\alpha, \beta \in \Delta, r, s, w \in R$,

$$\begin{aligned} (r \otimes_k s \otimes_k w) \otimes_{R^{\otimes 3}} (\alpha \otimes_R \beta) &= (\alpha \otimes_k \beta) \otimes_{R^{\otimes 3}} (r \otimes_k s \otimes_k w) \\ &= r \otimes_R \alpha \otimes_R s \otimes_R \beta \otimes_R w. \end{aligned} \quad (1.1-5)$$

Denote by $\mathbb{M}_{m \times n}(\Delta)$ the set of *matrices over Δ* of size $m \times n$; and by $\mathbb{T}_n(\Delta), \mathbb{N}_n(\Delta), \mathbb{D}_n(\Delta)$ the sets of upper triangular, strictly upper triangular, and diagonal Δ -matrices of size $n \times n$ respectively. The product of two Δ -matrices is the *usual matrix product*. If $H = (h_{ij}) \in \mathbb{M}_{m \times n}(R)$, $U = (u_{ij}) \in \mathbb{M}_{m \times n}(R \otimes_k R)$, $\alpha \in \Delta$, define

$$\begin{aligned} H \otimes_R \alpha &= (h_{ij} \otimes_R \alpha) \in \mathbb{M}_{m \times n}(\Delta), \\ \alpha \otimes_R H &= (\alpha \otimes_R h_{ij}) \in \mathbb{M}_{m \times n}(\Delta); \\ U \otimes_{R^{\otimes 2}} \alpha &= (\alpha \otimes_{R^{\otimes 2}} u_{ij}) = \alpha \otimes_{R^{\otimes 2}} U \in \mathbb{M}_{m \times n}(\Delta). \end{aligned} \quad (1.1-6)$$

The first two are based on Formula (1.1-1). For the last one, note that $(\alpha \otimes_{R^{\otimes 2}} U) \otimes_R H = \alpha \otimes_{R^{\otimes 2}} (UH), H \otimes_R (\alpha \otimes_{R^{\otimes 2}} U) = \alpha \otimes_{R^{\otimes 2}} (HU)$ by the note stated under Formula (1.1-2). Let $U = (u_{ij}) \in \mathbb{M}_{m \times n}(R), V = (v_{jl}) \in \mathbb{M}_{n \times r}(R \otimes_k R)$ and $\alpha, \beta \in \Delta$. Then

$$\begin{aligned} (\alpha \otimes_{R^{\otimes 2}} U)(\beta \otimes_{R^{\otimes 2}} V) &= (\sum_{j=1}^n ((\alpha \otimes_{R^{\otimes 2}} u_{ij}) \otimes_R (\beta \otimes_{R^{\otimes 2}} v_{jl})))_{i,l} \\ &= (\sum_{j=1}^n (\alpha \otimes_R \beta) \otimes_{R^{\otimes 3}} (u_{ij} \otimes_R v_{jl}))_{i,l} \\ &= (\alpha \otimes_R \beta) \otimes_{R^{\otimes 3}} (UV). \end{aligned} \quad (1.1-7)$$

An R - R -bimodule \mathcal{S}_1 is said to be a *quasi-free bimodule* finitely generated by U_1, \dots, U_m , provided that the morphism

$$(R_{X_1} \otimes_k R_{Y_1}) \oplus \cdots \oplus (R_{X_m} \otimes_k R_{Y_m}) \rightarrow \mathcal{S}_1, \quad 1_{X_i} \otimes_k 1_{Y_i} \mapsto U_i$$

is an isomorphism. In this case, $\{U_1, \dots, U_m\}$ is called an R - R -*quasi-free basis* of \mathcal{S}_1 , or R - R -quasi-basis of \mathcal{S}_1 for short.

Let $\mathcal{S}_p = R^{\otimes(p+1)} \otimes_{R^{\otimes 2}} \mathcal{S}_1$, which possesses an R - R -bimodule structure:

$$(r \otimes_k s) \otimes_{R^{\otimes 2}} (\alpha \otimes_{R^{\otimes 2}} U) = (r\alpha s) \otimes_{R^{\otimes 2}} U = \alpha \otimes_{R^{\otimes 2}} ((r \otimes_k s) \otimes_{R^{\otimes 2}} U) \quad (1.1-8)$$

for $r, s \in R, \alpha \in R^{p+1}, U \in \mathcal{S}$. Thus $\mathcal{S} = \sum_{p=1}^{\infty} \mathcal{S}_p = \bar{\Delta} \otimes_{R^{\otimes 2}} \mathcal{S}_1$ is an R - R -bimodule, and \mathcal{S}_p is said to have *index p* in \mathcal{S} .

Definition 1.1.1 Let $T = \{1, 2, \dots, t\}$ be a set of integers, and let \sim be an equivalent relation on T , such that the set T/\sim is one-to-one corresponding to the vertex set \mathcal{T} of a minimal algebra R . It may be written as $\mathcal{T} = T/\sim$.

Definition 1.1.2 (i) Define an R - R -bimodule:

$$\mathcal{K}_0 = \{\text{diag}(s_{11}, \dots, s_{tt}) \mid \text{when } i \in X, s_{ii} \in R_X, \text{ and } s_{ii} = s_{jj}, \forall i \sim j\}.$$

Let $E_X \in \mathbb{D}_t(R_X)$ with the entry $s_{ii} = 1_X$ if $i \in X$ and $s_{ii} = 0$ if $i \notin X$, then $\{E_X \mid X \in \mathcal{T}\}$ is an R -quasi-basis of \mathcal{K}_0 , and $E = \sum_{X \in \mathcal{T}} E_X$ is the identity matrix of size t .

(ii) Define a quasi-free R - R -bimodule $\mathcal{K}_1 \subseteq \mathbb{N}_t(R \otimes_k R)$ with an R - R -quasi-basis:

$$\mathcal{V} = \cup_{(X,Y) \in \mathcal{T} \times \mathcal{T}} \mathcal{V}_{XY} = \{V_1, V_2, \dots, V_m\}, \quad \mathcal{V}_{XY} \subset \mathbb{N}_t(R_X \otimes_k R_Y),$$

where $V \in \mathcal{V}_{XY}$ if and only if $1_X V 1_Y = V$.

(iii) Suppose $\mathcal{K} = \mathcal{K}_0 \oplus (\bar{\Delta} \otimes_{R^{\otimes 2}} \mathcal{K}_1)$ possesses an algebra structure, where the multiplication $m : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ is the usual matrix product over Δ consisting of $m_{pq} : \mathcal{K}_p \times \mathcal{K}_q \rightarrow \mathcal{K}_{p+q}, \forall p, q \geq 0$; the unit is given by the canonical inclusion $e : R \cong \mathcal{K}_0 \hookrightarrow \mathcal{K}$. \mathcal{K} is said to be *finitely generated in index (0, 1) over Δ* .

Clearly $\mathcal{K}_0 \simeq R$ as algebras. The multiplication $m_{pq}, \forall p, q > 0$ is determined by $m_{11} : \mathcal{K}_1 \times \mathcal{K}_1 \rightarrow \mathcal{K}_2$, see Formula (1.1-7). And for any $r \in R, \alpha \in \Delta, (r \otimes_R E_X) \otimes_R (\alpha \otimes_{R^{\otimes 2}} V_j) = (r \otimes_R \alpha) \otimes_{R^{\otimes 2}} V_j$ if $1_X V_j = V_j$, or 0 otherwise, $(\alpha \otimes_{R^{\otimes 2}} V_j) \otimes_R (r \otimes_R E_X)$ is similar. $\{E_X \mid X \in \mathcal{T}\}$ is a set of orthogonal primitive idempotents of \mathcal{K} , and $E = e(1_R)$ is the *identity*.

Let $T = \{1, 2, \dots, t\}$ and $T' = \{1, 2, \dots, t'\}$ be two sets of integers. An *order* on $T \times T'$ is defined as follows: $(i, j) \preceq (i', j')$ provided that $i > i'$, or $i = i'$ but $j \leq j'$. Thus an order on the index set of the entries of a matrix in $\text{IM}_{t \times t'}(\Delta)$ is obtained. Let $M = (\lambda_{ij}) \in \text{IM}_{t \times t'}(\Delta)$. The entry λ_{pq} is said to be the *leading entry* of M if $\lambda_{pq} \neq 0$, and any $\lambda_{ij} \neq 0$ implies that $(p, q) \preceq (i, j)$. Let $\bar{M} = (C_{ij})$ be a partitioned matrix over Δ , one defines similarly the *leading block* of \bar{M} . In both cases, the index (p, q) is called the *leading position* of M resp. \bar{M} .

Let \mathcal{S} be a subspace of $\text{IM}_t(k)$. An ordered basis $\mathcal{U} = \{U_1, \dots, U_r\}$ with the leading positions $(p_1, q_1), \dots, (p_r, q_r)$ respectively is called a *normalized basis* of \mathcal{S} provided that

- (i) the leading entry of U_i is 1;
- (ii) the (p_i, q_i) -th entry of U_j is 0 for $j \neq i$;
- (iii) $U_i \preceq U_j$ if and only if $(p_i, q_i) \preceq (p_j, q_j)$

The basis \mathcal{U} is a linearly ordered set. It is easy to see that \mathcal{S} has a normalized basis by Linear algebra. In fact, if t^2 variables x_{ij} under the order of matrix indices defined as above are taken, then \mathcal{S} will be the solution space of some system of linear equations $\sum_{(i,j) \in T \times T} a_{ij}^l x_{ij} = 0, a_{ij}^l \in k, 1 \leq l \leq s$ for some positive integer s . Reducing the coefficient matrix to the simplest echelon form, we assume that $x_{p_1 q_1}, \dots, x_{p_r q_r}$ are all the free variables, and $\{U_1, \dots, U_r\}$ is a basic system of solutions, whose (p_i, q_i) -entry is 1 and $(p, q) \prec (p_i, q_i)$ -entry is 0 for $i = 1, \dots, r$, a normalized basis \mathcal{U} of \mathcal{S} is obtained.

Definition 1.1.3 (i) Define a quasi-free R - R -bimodule $\mathcal{M}_1 \subseteq \text{IM}_t(R \otimes_k R)$, such that $E_X \mathcal{M}_1 E_Y$ has a normalized quasi-basis $\mathcal{A}_{XY} \subseteq \text{IM}_t(k 1_X \otimes_k 1_Y k) \simeq \text{IM}_t(k)$ as k -vector spaces, where $A \in \mathcal{A}_{XY}$ if and only if $1_X A 1_Y = A$. Thus there is a *normalized quasi-basis*:

$$\mathcal{A} = \cup_{(X,Y) \in \mathcal{T} \times \mathcal{T}} \mathcal{A}_{XY} = \{A_1, A_2, \dots, A_n\}.$$

(ii) Let $\mathcal{M} = \bar{\Delta} \otimes_{R^{\otimes 2}} \mathcal{M}_1$, and the algebra \mathcal{K} be given by Definition 1.1.2. Define a \mathcal{K} - \mathcal{K} -bimodule structure on \mathcal{M} , such that the left module action $l : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{M}$ consists of $l_{pq} : \mathcal{K}_p \times \mathcal{M}_q \rightarrow \mathcal{M}_{p+q}, \forall p \geq 0, q > 0$, and the right one $r : \mathcal{M} \times \mathcal{K} \rightarrow \mathcal{M}$ consists of $r_{pq} : \mathcal{M}_p \times \mathcal{K}_q \rightarrow \mathcal{M}_{p+q}, \forall p > 0, q \geq 0$ given by usual matrix product respectively. The \mathcal{K} - \mathcal{K} -bimodule \mathcal{M} is said to be *finitely generated in index (0, 1) with $\mathcal{M}_0 = \{0\}$* .

Definition 1.1.4 Let $H = \sum_{X \in \mathcal{T}} H_X \in \text{IM}_t(R)$ be a matrix, where $H_X = (h_{ij})_{t \times t} \in E_X \text{IM}(R) E_X$ with $h_{ij} \in R_X$ for $i, j \in X$, and $h_{ij} = 0$ otherwise. Define a derivation $d : \mathcal{K} \rightarrow \mathcal{M}, U \mapsto UH - HU$, yielded by H consists of $d_p : \mathcal{K}_p \rightarrow \mathcal{M}_p, \forall p \geq 0$.

It is not difficult to see that $d_0 = 0$, and d_p is determined by d_1 for $p > 0$ according to the note stated under Formula (1.1-6).

Definition 1.1.5 A quadruple $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is called a *matrix bimodule problem* provided

- (i) R is a minimal algebra with a vertex set \mathcal{T} given by Definition 1.1.1;
- (ii) \mathcal{K} is an algebra given by Definition 1.1.2;
- (iii) \mathcal{M} is a \mathcal{K} - \mathcal{K} -bimodule given by Definition 1.1.3;
- (iv) There is a derivation $d : \mathcal{K} \rightarrow \mathcal{M}$ given by Definition 1.1.4.

In particular, if $\mathcal{M} = 0$, \mathfrak{A} is said to be a *minimal matrix bimodule problem*.

1.2 Bi-comodule problems and Bocses

We define a notion of bi-comodule problems associated to matrix bimodule problems, which is the transition into bocses. The concepts and the proofs are proposed by Y. Han.

Since \mathcal{K}_1 and \mathcal{M}_1 are both quasi-free R - R -bimodules, we have their $R^{\otimes 2}$ -dual structures \mathcal{C}_1 and \mathcal{N}_1 with R - R -quasi-basis \mathcal{V}^* and \mathcal{A}^* respectively:

$$\begin{aligned} \mathcal{C}_1 &= \text{Hom}_{R^{\otimes 2}}(\mathcal{K}_1, R^{\otimes 2}), & \mathcal{V}^* &= \{v_1, v_2, \dots, v_m\}; \\ \mathcal{N}_1 &= \text{Hom}_{R^{\otimes 2}}(\mathcal{M}_1, R^{\otimes 2}), & \mathcal{A}^* &= \{a_1, a_2, \dots, a_n\}. \end{aligned} \quad (1.2-1)$$

Write $v : X \mapsto Y$ (resp. $a : X \mapsto Y$) provided that $V \in \mathcal{V}_{XY}$ (resp. $A \in \mathcal{A}_{XY}$).

The quasi-basis \mathcal{V} of \mathcal{K}_1 has a natural partial order, namely, $V_i \prec V_j$, if their leading positions $(p_i, q_i) \prec (p_j, q_j)$. Thus $V_i V_j = \sum_{l < i, j} \gamma_{ijl} \otimes_{R^{\otimes 2}} V_l$, since $\mathcal{V} \subseteq \mathbb{N}_t(R \otimes_k R)$. For a fixed pair (p, q) , any fixed order on the set $\{V_i \mid (p_i, q_i) = (p, q)\}$ may be defined, which gives a linear order on \mathcal{V} .

Definition 1.2.1 Let \mathcal{K} be a k -algebra as in Definition 1.1.2. We define a quasi-free R -module $\mathcal{C}_0 = \text{Hom}_R(\mathcal{K}_0, R) \simeq \sum_{X \in \mathcal{T}} R_X e_X \simeq R$ with an R -quasi-basis $\{e_X\}_{X \in \mathcal{T}}$ dual to $\{E_X\}_{X \in \mathcal{T}}$ of \mathcal{K}_0 ; and a quasi-free R - R -bimodule \mathcal{C}_1 with an R - R -quasi-basis \mathcal{V}^* defined by the first formula of (1.2-1), which has a linear order yielded from that of \mathcal{V} . Write $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, and define a coalgebra structure with a counit $\varepsilon : \mathcal{C} \rightarrow R$, $e_X \mapsto 1_X$, $v_j \mapsto 0$ and a comultiplication $\mu : \mathcal{C} \mapsto \mathcal{C} \otimes_R \mathcal{C}$ dual to $(m_{00}, m_{01}, m_{10}, m_{11})$:

$$\begin{aligned} \mu &= \begin{pmatrix} \mu_{00} \\ \mu_{10} + \mu_{01} + \mu_{11} \end{pmatrix} : \begin{pmatrix} \mathcal{C}_0 \\ \mathcal{C}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{C}_0 \otimes_R \mathcal{C}_0, \\ \mathcal{C}_1 \otimes_R \mathcal{C}_0 \oplus \mathcal{C}_0 \otimes_R \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes_R \mathcal{C}_1 \end{pmatrix} \\ \mu_{00}(e_X) &= e_X \otimes_R e_X, \quad \mu_{10}(v_l) = v_l \otimes_R e_{t(v_l)}, \\ \mu_{01}(v_l) &= e_{s(v_l)} \otimes_R v_l, \quad \mu_{11}(v_l) = \sum_{i, j > l} \gamma_{ijl} \otimes_{R^{\otimes 3}} (v_i \otimes_R v_j). \end{aligned}$$

Since \mathcal{A} is linearly ordered and $\mathcal{V} \subset \mathbb{N}_t(R \otimes_k R)$, $V_i A_j = \sum_{l > j} \eta_{ijl} \otimes_{R^{\otimes 2}} A_l$ and $A_i V_j = \sum_{l > i} \sigma_{ijl} \otimes_{R^{\otimes 2}} A_l$.

Definition 1.2.2 Let \mathcal{M} be a \mathcal{K} - \mathcal{K} -bimodule as in Definition 1.1.3. A quasi free R - R -bimodule \mathcal{N}_1 with an R - R -quasi-basis \mathcal{A}^* given by the second formula of (1.2-1) is define. Write $\mathcal{N} = \mathcal{N}_1$, then \mathcal{N} has a \mathcal{C} - \mathcal{C} -bi-comodule structure with the left and right co-module actions dual to (l_{01}, l_{11}) and (r_{10}, r_{11}) respectively:

$$\begin{aligned} \iota &= (l_{01} + l_{11}) : \mathcal{N} \rightarrow \mathcal{C} \otimes_R \mathcal{N} = \mathcal{C}_0 \otimes_R \mathcal{N} \oplus \mathcal{C}_1 \otimes_R \mathcal{N}, \\ l_{01}(a_l) &= e_{s(a_l)} \otimes_R a_l, \quad l_{11}(a_l) = \sum_{j < l, i} \eta_{ijl} \otimes_{R^{\otimes 3}} (v_i \otimes_R a_j); \\ \tau &= (r_{10} + r_{11}) : \mathcal{N} \rightarrow \mathcal{N} \otimes_R \mathcal{C} = \mathcal{N} \otimes_R \mathcal{C}_0 \oplus \mathcal{N} \otimes_R \mathcal{C}_1, \\ r_{10}(a_l) &= a_l \otimes_R e_{t(a_l)}, \quad r_{11}(a_l) = \sum_{i < l, j} \sigma_{ijl} \otimes_{R^{\otimes 3}} (a_i \otimes_R v_j). \end{aligned}$$

Definition 1.2.3 Assume $d_1(V_i) = \sum_l \zeta_{il} \otimes_{R^{\otimes 2}} A_l$ defined in 1.1.4. There is a co-derivation $\partial = (\partial_0, \partial_1) : \mathcal{N} \rightarrow \mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ with $\partial_0 = 0$ and $\partial_1(a_l) = \sum_i \zeta_{il} \otimes_{R^{\otimes 2}} v_i$ dual to (d_0, d_1) , such that $\mu\partial = (\mathbb{1} \otimes \partial)\iota + (\partial \otimes \mathbb{1})\tau$.

Definition 1.2.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem. A quadruple $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ is said to be a *bi-comodule problem associated to \mathfrak{A}* provided

- (i) R is a minimal algebra with a vertex set \mathcal{T} ;
- (ii) \mathcal{C} is a co-algebra given by Definition 1.2.1;
- (iii) \mathcal{N} is a \mathcal{C} - \mathcal{C} -bi-comodule given by Definition 1.2.2;
- (iv) $\partial : \mathcal{N} \rightarrow \mathcal{C}$ is a co-derivation given by Definition 1.2.3.

Now we construct a bocs via the bi-comodule problem \mathfrak{C} associated to \mathfrak{A} . Write $\mathcal{N}^{\otimes p} = \mathcal{N} \otimes_R \dots \otimes_R \mathcal{N}$ with p copies of \mathcal{N} and $\mathcal{N}^{\otimes 0} = R$. Define a tensor algebra Γ of \mathcal{N} over R , whose multiplication is given by the natural isomorphisms:

$$\Gamma = \bigoplus_{p=0}^{\infty} \mathcal{N}^{\otimes p}; \quad \mathcal{N}^{\otimes p} \otimes_R \mathcal{N}^{\otimes q} \simeq \mathcal{N}^{\otimes(p+q)}.$$

Let $\Xi = \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma$ be a Γ - Γ -bimodule of co-algebra structure induced by $R \hookrightarrow \Gamma$, and denoted by $(\Xi, \mu_\Xi, \varepsilon_\Xi)$. Define the following three R - R -bimodule maps:

$$\begin{aligned}\kappa_1 : \mathcal{N} &\xrightarrow{\iota} \mathcal{C} \otimes_R \mathcal{N} \xrightarrow{\cong} R \otimes_R \mathcal{C} \otimes_R \mathcal{N} \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\ \kappa_2 : \mathcal{N} &\xrightarrow{\tau} \mathcal{N} \otimes_R \mathcal{C} \xrightarrow{\cong} \mathcal{N} \otimes_R \mathcal{C} \otimes_R R \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma, \\ \kappa_3 : \mathcal{N} &\xrightarrow{\partial} \mathcal{C} \xrightarrow{\cong} R \otimes_R \mathcal{C} \otimes_R R \hookrightarrow \Gamma \otimes_R \mathcal{C} \otimes_R \Gamma.\end{aligned}$$

Lemma 1.2.5 $\text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a Γ -coideal in Ξ . Thus $\Omega := \Xi / \text{Im}(\kappa_1 - \kappa_2 + \kappa_3)$ is a Γ - Γ -bimodule of coalgebra structure.

Proof Recall the law of bi-comodule: $(\mu \otimes \mathbb{1})\iota = (\mathbb{1} \otimes \iota)\iota$, $(\mathbb{1} \otimes \mu)\tau = (\tau \otimes \mathbb{1})\tau$, $(\mathbb{1} \otimes \tau)\iota = (\iota \otimes \mathbb{1})\tau$ and $(\mathbb{1} \otimes \partial)\iota - \mu\partial + (\partial \otimes \mathbb{1})\tau = 0$. Thus, for any $b \in \mathcal{N}$, we have

$$\begin{aligned}&\mu_\Xi(\kappa_1 - \kappa_2 + \kappa_3)(b) \\&= \mu_\Xi(1_\Gamma \otimes \iota(b)) - \mu_\Xi(\tau(b) \otimes 1_\Gamma) + \mu_\Xi(\mathbb{1} \otimes \partial(b) \otimes 1_\Gamma) \\&= (\mu \otimes \mathbb{1})\iota(b) - (\mathbb{1} \otimes \mu)\tau(b) + \mu(\partial(b)) \\&= (\mathbb{1} \otimes \iota)\iota(b) - (\tau \otimes \mathbb{1})\tau(b) + (\mathbb{1} \otimes \partial)\iota(b) + (\partial \otimes \mathbb{1})\tau(b) \\&= (\mathbb{1} \otimes \iota - \mathbb{1} \otimes \tau + \mathbb{1} \otimes \partial)\iota(b) + (\iota \otimes \mathbb{1} - \tau \otimes \mathbb{1} + \partial \otimes \mathbb{1})\tau(b) \\&= u_{(1)} \otimes (\iota - \tau + \partial)(b_{(1)}) + (\iota - \tau + \partial)(b_{(2)}) \otimes u_{(2)} \\&= u_{(1)} \otimes (\kappa_1 - \kappa_2 + \kappa_3)(b_{(1)}) + (\kappa_1 - \kappa_2 + \kappa_3)(b_{(2)}) \otimes u_{(2)} \\&\in \Xi \otimes \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) + \text{Im}(\kappa_1 - \kappa_2 + \kappa_3) \otimes \Xi\end{aligned}$$

where $\iota(b) := u_{(1)} \otimes b_{(1)}$, $\tau(b) := b_{(2)} \otimes u_{(2)}$, and each term in each step is viewed as an element in $\Xi \otimes_\Gamma \Xi$ naturally. \square

Recall from [Ro] and [CB1], $\mathfrak{B} = (\Gamma, \Omega)$ defined as above is a bocs with a layer

$$L = (R; \omega; a_1, a_2, \dots, a_n; v_1, v_2, \dots, v_m).$$

Denote by ε_Ω and μ_Ω the induced co-unit and co-multiplication, then $\bar{\Omega} = \ker \varepsilon_\Omega$ is a Γ - Γ -bimodule freely generated by v_1, v_2, \dots, v_m , and $\Omega = \Gamma \oplus \bar{\Omega}$ as bimodules. From this, we use the embedding: $\mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{N} \otimes \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes \mathcal{N} \hookrightarrow \Gamma \otimes_R (\mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{N} \otimes_R \mathcal{C}_1 \oplus \mathcal{C}_1 \otimes_R \mathcal{N}) \otimes_R \Gamma \subset \Omega$; and the isomorphism: $\bar{\Omega} \otimes_R \bar{\Omega} \simeq \bar{\Omega} \otimes_\Gamma \bar{\Omega}$.

The group-like $\omega : R \rightarrow \Omega, 1_X \mapsto e_X$ is an R - R -bimodule map. Recall from [CB1, 3.3 Definition], and note that $(\iota_0(a_i) + \iota_1(a_i)) - (\tau_0(a_i) + \tau_1(a_i)) + \partial_1(a_i) = (\kappa_1 - \kappa_2 + \kappa_3)(a_i) = 0$ in Ω , the pair of the differentials determined by ω is given by $\delta_1 : \Gamma \rightarrow \bar{\Omega}$:

$$\begin{aligned}\delta_1(1_X) &= 1_X e_X - e_X 1_X = 0, \quad X \in \mathcal{T}, \\ \delta_1(a_i) &= a_i \otimes_R e_{t(a_i)} - e_{s(a_i)} \otimes_R a_i = \tau_0(a_i) - \iota_0(a_i) \\ &= \iota_1(a_i) - \tau_1(a_i) + \partial_1(a_i), \quad 1 \leq j \leq n;\end{aligned}$$

and $\delta_2 : \bar{\Omega} \mapsto \bar{\Omega} \otimes_\Gamma \bar{\Omega}, \delta_2(v_j) = \mu(v_j) - e_{s(v_j)} \otimes_R v_j - v_j \otimes_R e_{t(v_j)} = \mu_{11}(v_j), 1 \leq j \leq m$. Then the bocs \mathfrak{B} is said to be the *bocs associated to the matrix bimodule problem* \mathfrak{A} . Denote by $(\mathfrak{A}, \mathfrak{B})$ the pair of a matrix bimodule problem and its associated bocs, or just the pair $(\mathfrak{A}, \mathfrak{B})$.

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the associated bi-comodule problem \mathfrak{C} . Then the module actions

$$l(\mathcal{K}_1 \times A_i), r(A_i \times \mathcal{K}_1) \subseteq \oplus_{l=i+1}^n R^{\otimes 3} \otimes_{R^{\otimes 2}} A_l \quad (1.2-2)$$

by the fact stated before Definition 1.2.2, which is called the *triangular property*. The left and the right co-module actions also possess the *triangular property* by 1.2.2:

$$\iota_1(a_l) \in \oplus_{i=1}^{l-1} \mathcal{C}_1 \otimes_R a_i, \quad \tau_1(a_l) \in \oplus_{i=1}^{l-1} a_i \otimes_R \mathcal{C}_1. \quad (1.2-3)$$

Define a \mathcal{K} - \mathcal{K} sub-bimodule of \mathcal{M} , then a \mathcal{K} - \mathcal{K} -quotient-bimodule of \mathcal{M} :

$$\mathcal{M}^{(h)} = \oplus_{i=h+1}^n \bar{\Delta} \otimes_{R \otimes 2} A_i \subseteq \mathcal{M}, \quad \mathcal{M}^{[h]} = \mathcal{M} / \mathcal{M}^{(h)}.$$

$\mathfrak{A}^{[h]} = (R, \mathcal{K}, \mathcal{M}^{[h]}, \bar{d})$ with \bar{d} induced from d is said to be a *quotient problem* of \mathfrak{A} , but $\mathfrak{A}^{[h]}$ itself might be no longer a matrix bimodule problem. If $\mathcal{N}^{(h)} = \oplus_{i=1}^h R \otimes_R a_i \otimes_R R$, then $\mathfrak{C}^{(h)} = (R, \mathcal{C}, \mathcal{N}^{(h)}, \partial|_{\mathcal{N}^{(h)}})$ is a sub-bi-comodule problem of \mathfrak{C} . If $\Gamma^{(h)}$ is a tensor algebra freely generated by a_1, \dots, a_h , then the boc $\mathfrak{B} = (\Gamma, \Omega)$ has a sub-boc $\mathfrak{B}^{(h)} = (\Gamma^{(h)}, \Gamma^{(h)} \otimes_R \Omega \otimes_R \Gamma^{(h)})$.

Note a simple fact: let $(\mathfrak{A}, \mathfrak{C}, \mathfrak{B})$ be a triple defined as above, then

$$\begin{aligned} & l(\mathcal{K}_1 \times \mathcal{M}_1), r(\mathcal{M}_1 \times \mathcal{K}_1), d(\mathcal{K}_1) \subseteq \mathcal{M}_1^{(h)} \text{ in } \mathfrak{A} \\ \iff & \mathcal{C}_1 \otimes_R \mathcal{N}_1^{(h)} = 0, \mathcal{N}_1^{(h)} \otimes_R \mathcal{C}_1 = 0, \partial(\mathcal{N}_1^{(h)}) = 0 \text{ in } \mathfrak{C} \\ \iff & \delta(\Gamma^{(h)}) = 0 \text{ in } \mathfrak{B}. \end{aligned} \quad (1.2-4)$$

In fact, the condition in \mathfrak{A} is equivalent to $\eta_{jil} = 0, \sigma_{ijl} = 0, \zeta_{jl} = 0$ for $l = 1, \dots, h$ and any i, j , which is equivalent to the conditions in \mathfrak{C} and \mathfrak{B} .

Recall from [CB1] that a representation of a layered boc \mathfrak{B} is a left Γ -module P of dimension vector \underline{d} consisting of three sets:

$$\begin{aligned} & \{P_X = k^{d_X} \mid X \in \mathcal{T}\}, \quad \{P(x) : P_X \rightarrow P_X \mid X \in \mathcal{T}\}; \\ & \{P(a_i) : k^{d_{X_i}} \rightarrow k^{d_{Y_i}} \mid a_i : X_i \rightarrow Y_i, i = 1, \dots, n\}. \end{aligned} \quad (1.2-5)$$

A morphism from P to Q is given by a Γ -map $f : \Omega \otimes_\Gamma P \rightarrow Q$. Clearly, $\text{Hom}_\Gamma(\bar{\Omega} \otimes_\Gamma P, Q) \simeq \oplus_{j=1}^m \text{Hom}_\Gamma(\Gamma 1_{s(v_j)} \otimes_k 1_{t(v_j)} P, Q) \simeq \oplus_{j=1}^m \text{Hom}_k(1_{t(v_j)} P, 1_{s(v_j)} Q)$. Write

$$f = \{f_X; f(v_j) \mid X \in \mathcal{T}, 1 \leq j \leq m\}, \quad (1.2-6)$$

then [BK] shows that f is a morphism if and only if for all $a_l \in \mathcal{A}^*, 1 \leq l \leq n$:

$$\begin{aligned} P(a_l) f_{Y_l} - f_{X_l} Q(a_l) &= \sum_{j < l, i} \eta_{ijl} \otimes_{R \otimes 3} (f(v_i) \otimes_R Q(a_j)) \\ &\quad - \sum_{i < l, j} \sigma_{ijl} \otimes_{R \otimes 3} (P(a_i) \otimes_R f(v_j)) + \sum_i \zeta_{il} \otimes_{R \otimes 2} f(v_i). \end{aligned} \quad (1.2-7)$$

1.3 Representation categories of matrix bimodule problems

In this subsection, a notion of “*-product” and the operations between *-products are defined, which will be used frequently throughout the paper. Based on this nation, the representation category of a matrix bimodule problem is defined. It is relatively complicated, but seems to be useful for the proof of the main theorem.

Definition 1.3.1 Let $J(\lambda) = J_d(\lambda)^{e_d} \oplus J_{d-1}(\lambda)^{e_{d-1}} \oplus \dots \oplus J_1(\lambda)^{e_1}$ be a Jordan form, where e_i are non-negative integers. Denote $e_d + e_{d-1} + \dots + e_j$ by m_j for $j = 1, \dots, d$. The following partitioned matrix $W(\lambda)$ similar to $J(\lambda)$ is called a *Weyr matrix of eigenvalue λ* :

$$W(\lambda) = \begin{pmatrix} \lambda I_{m_1} & W_{12} & 0 & \cdots & 0 & 0 \\ & \lambda I_{m_2} & W_{23} & \cdots & 0 & 0 \\ & & \lambda I_{m_3} & \cdots & 0 & 0 \\ & & & \ddots & \vdots & \vdots \\ & & & & \lambda I_{m_{d-1}} & W_{d-1,d} \\ & & & & & \lambda I_{m_d} \end{pmatrix}_{d \times d},$$

where $W_{j,j+1} = (I_{m_{j+1}} 0)^T$ of size $m_j \times m_{j+1}$ with superscript T denoting transpose. A direct sum $W = W(\lambda_1) \oplus W(\lambda_2) \oplus \cdots \oplus W(\lambda_s)$ with distinct eigenvalues λ_i is said to be a *Weyr matrix*. An order “ \prec ” on the base field k may be defined, so that each Weyr matrix has a unique form. Similarly, let $\{Z_{ij} \mid i, j \in \mathbb{Z}^+\}$ be a finite set of vertices, and $S = \oplus_{i,j} k 1_{Z_{ij}} \oplus k[z, \phi(z)^{-1}] 1_Z$ be a minimal algebra. $\bar{W} \simeq \oplus J_{ij}(\lambda_i)^{e_{ij}} 1_{Z_{ij}}$ or $\oplus J_{ij}(\lambda_i)^{e_{ij}} 1_{Z_{ij}} \oplus (z 1_Z)$ with $\{e_{ij}\} \subset \mathbb{Z}^+$ is said to be a *Weyr matrix over S*. It is possible that some summands of \bar{W} are diagonal blocks with the diagonal entries being the primitive idempotents of S .

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem having a set of integers $T = \{1, 2, \dots, t\}$ with partition \mathcal{T} . A Weyr matrix W over k is called R_X -regular for $X \in \mathcal{T}_1$, if all the eigenvalues λ of W have the property that $\phi_X(\lambda) \neq 0$. An identity matrix I is also called an R_X -regular Weyr matrix for $X \in \mathcal{T}_0$. A vector of non-negative integers is said to be a *size vector* $\underline{m} = (m_1, m_2, \dots, m_t)$ over \mathcal{T} , if $m_i = m_j, \forall i \sim j$. And $\sum_{i=1}^t m_i$ is called the *size* of \underline{m} .

Definition 1.3.2 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem, S a minimal algebra, and $\Sigma = \oplus_{p=1}^{\infty} S^{\otimes p}$, see Formula (1.1-3).

(i) Write $H_X = (h_{ij}(x) 1_X)_{t \times t}$ with $h_{ij}(x) \in k[x]$ for $X \in \mathcal{T}_1$, and $x = 1, h_{ij}(x) \in k$ for $X \in \mathcal{T}_0$. Let \bar{W}_X be a Weyr matrix of size m_X over S . There exists an $\underline{m} \times \underline{m}$ -partitioned matrix over S :

$$H_X(\bar{W}_X) = (B_{ij})_{t \times t}, \quad B_{ij} = \begin{cases} h_{ij}(\bar{W}_X)_{m_i \times m_j}, & i, j \in X, \\ (0)_{m_i \times m_j}, & i \notin X \text{ or } j \notin X, \end{cases}$$

(ii) Let $\underline{m} = (m_1, \dots, m_t)$ and $\underline{n} = (n_1, \dots, n_t)$ be two size vectors over \mathcal{T} , and let $F \in \text{IM}_{m_X \times n_X}(S^{\otimes p}), p = 1, 2$, with an R_X - R_X -module structure. The star product $*$ of F_X and E_X is defined to be a diagonal $\underline{m} \times \underline{n}$ -partitioned matrix:

$$F_X * E_X = \text{diag}(B_{11}, \dots, B_{tt}), \quad B_{ii} = \begin{cases} F_X, & i \in X, \\ 0 & i \notin X, \end{cases}$$

(iii) Let $U = (u_{ij}) \in \mathcal{V}_{XY} \cup \mathcal{A}_{XY}$, and $\{\bar{W}_X \in \text{IM}_{m_X}(S) \mid X \in \mathcal{T}\}, \{\bar{W}'_Y \in \text{IM}_{n_Y}(S) \mid Y \in \mathcal{T}\}$ be two sets of regular Weyr matrices. Suppose there is an R_X - R_Y -bimodule structure on $\text{IM}_{m_X \times n_Y}(S^{\otimes p})$ for $p = 1, 2$, and $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes p})$:

$$C \otimes_{R^{\otimes 2}} (x \otimes_k y) = W_X C W'_Y. \quad (1.3-1)$$

The star product $*$ of C and U is defined to be an $(\underline{m} \times \underline{n})$ -partitioned matrix:

$$C * U = (B_{ij})_{t \times t}, \quad B_{ij} = \begin{cases} C \otimes_{R^{\otimes 2}} u_{ij}, & i \in X, j \in Y; \\ (0)_{m_i \times n_j}, & i \notin X, \text{ or } j \notin Y. \end{cases}$$

Lemma 1.3.3 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem.

(i) If $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes 2})$, $d(V_i) = \sum_l \zeta_{il} A_l$ with $\zeta_{il} \in R_X \otimes_k R_Y$ are given by Definition 1.2.3, then by the usual product of Σ -matrices:

$$(C * V_i) H_Y(\bar{W}_Y) - H_X(\bar{W}_X)(C * V_i) = \sum_{l=1}^n (C \otimes_{R^{\otimes 2}} \zeta_{il}) * A_l.$$

(ii) If $F_X \in \text{IM}_{m_X \times n_X}(S^{\otimes p}), p = 1, 2$ and $C \in \text{IM}_{m_X \times n_Y}(S^{\otimes q}), q = 1, 2$, then by the usual product of Σ -matrices:

$$(F_X * E_X)(C * U) = \begin{cases} (F_X C) * U, & 1_X U = U; \\ 0, & 1_X U = 0. \end{cases}$$

Similarly, $(C * U)(F_X * E_X) = (C F_X) * U$ for $U 1_X = U$ and 0 otherwise. Moreover,

$$(F_X * E_X)(F'_X * E_X) = (F_X F'_X) * E_X, \quad F_X, F'_X \in \text{IM}_{m_X}(S^{\otimes p}), p = 1, 2.$$

(iii) Let $U \in E_X \mathbb{M}_t(R \otimes_k R) E_Y, V \in E_Y \mathbb{M}_t(R \otimes_k R) E_Z, G_l \in E_X \mathbb{M}_t(R \otimes_k R) E_Z$, where $UV = \sum_{l=1}^n \epsilon_l \otimes_{R^{\otimes 2}} G_l$, $\epsilon_l \in R_X \otimes_k R_Y \otimes_k R_Z$. Let $\underline{m}, \underline{n}, l$ be size vectors over \mathcal{T} , $C \in \mathbb{M}_{m_X \times n_Y}(S^{\otimes p}), D \in \mathbb{M}_{n_Y \times l_Z}(S^{\otimes q})$ for $(p, q) \in \{(2, 2), (1, 2), (2, 1)\}$. Then by the usual Σ -matrix product:

$$(C * U)(D * V) = \sum_{l=1}^n ((C \otimes_R D) \otimes_{R^{\otimes 3}} \epsilon_l) * G_l,$$

where the tensor product $\oplus_{(X,Y,Z) \in \mathcal{T} \times \mathcal{T} \times \mathcal{T}} \mathbb{M}_{m_X n_Y}(S^{\otimes p}) \otimes_R \mathbb{M}_{n_Y \times l_Z}(S^{\otimes q})$ has an $R^{\otimes 3}$ -module structure yielded from the R - R -bimodule structures given by Formula (1.3-1).

Proof (i) Write $H_X = (\alpha_{pq}), \alpha_{pq} \in R_X; H_Y = (\beta_{pq}), \beta_{pq} \in R_Y; V_i = (v_{pq}), v_{pq} \in R_X \otimes_k R_Y$,

$$\begin{aligned} \text{The left side} &= (\sum_l (C \otimes_{R^{\otimes 2}} v_{pl}) \otimes_R \beta_{lq}) - (\sum_l \alpha_{pl} \otimes_R (C \otimes_{R^{\otimes 2}} v_{lq})) \\ &= (C \otimes_{R^{\otimes 2}} \sum_l (v_{pl} \beta_{lq} - \alpha_{pl} v_{lq})) = C * (V_i H_Y - H_X V_i) \\ &= C * d(V_i) = C * (\sum_l \zeta_{il} A_l) = \text{the right side.} \end{aligned}$$

(ii) Write $U = (u_{pq}), u_{pq} \in R_X \otimes_k R_Y$, the left side $= (F_X 1_X (C \otimes_{R^{\otimes 2}} u_{pq})) = ((F_X C) \otimes_{R^{\otimes 2}} u_{pq}) = \text{the right side.}$

(iii) Write $U = (u_{pq}), u_{pq} \in R_X \otimes_k R_Y, V = (v_{pq}), v_{pq} \in R_Y \otimes_k R_Z$. The left side $= (\sum_l (C \otimes_{R^{\otimes 2}} u_{pl}) (D \otimes_{R^{\otimes 2}} v_{lq})) = ((CD) \otimes_{R^{\otimes 3}} (\sum_l u_{pl} \otimes_R v_{lq})) = \text{the right side.} \quad \square$

Definition 1.3.4 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem, and \underline{m} a size vector over \mathcal{T} . Thus a representation \bar{P} of \mathfrak{A} can be written as an $\underline{m} \times \underline{m}$ -partitioned matrix over k :

$$\bar{P} = \sum_{X \in \mathcal{T}} H_X(W_X) + \sum_{i=1}^n P(a_i) * A_i,$$

where $W_X \in \mathbb{M}_{m_X}(k)$ is an R_X -regular Weyr matrix for any $X \in \mathcal{T}$, and $P(a_i) \in \mathbb{M}_{m_{X_i} \times m_{Y_i}}(k)$. Taken $S = k$, the first summand is defined in 1.3.2 (i), and the second one in (iii).

Definition 1.3.5 Let \bar{P}, \bar{Q} be two representations of size vectors $\underline{m}, \underline{n}$ respectively. A morphism $\bar{f} : \bar{P} \rightarrow \bar{Q}$ is an $\underline{m} \times \underline{n}$ -partitioned matrix obtained from Definition 1.3.2 (ii) and (iii) for $S = k$:

$$\bar{f} = \sum_{X \in \mathcal{T}} f_X * E_X + \sum_{j=1}^m f(v_j) * V_j,$$

where $f_X \in \mathbb{M}_{m_X \times n_X}(k), f(v_j) \in \mathbb{M}_{m_{s(v_j)} \times n_{t(v_j)}}(k)$, such that $\bar{P} \bar{f} = \bar{f} \bar{Q}$ under the matrix product given according to Lemma 1.3.3 (i)–(iii).

If \bar{U} is an object and $\bar{f}' : \bar{Q} \rightarrow \bar{U}$ a morphism, then $\bar{f} \bar{f}' : \bar{P} \rightarrow \bar{U}$ calculated according to Lemma 1.3.3 (ii)–(iii) is still a morphism. In fact, $(\bar{f} \bar{f}') \bar{P} = \bar{f}(\bar{Q} \bar{f}') = (\bar{U} \bar{f}) \bar{f}' = \bar{U}(\bar{f} \bar{f}')$. We denote by $R(\mathfrak{A})$ the representation category of the matrix bimodule problem \mathfrak{A} .

1.4. Formal Products and Formal Equations

In this subsection, 1) a concept of “formal equation” is introduced to build up a nice connection between a matrix bimodule problem and its associated boc; and 2) a special class of bipartite matrix bimodule problems is noticed, because of the close relation between such problems and finite dimensional algebras.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem, with the associated bi-comodule problem $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ and the boc \mathfrak{B} . Recall that $\{E_X\}$ and $\{e_X\}$ are dual bases of $(\mathcal{K}_0, \mathcal{C}_0)$; $\{V_1, \dots, V_m\}$ and $\{v_1, \dots, v_m\}$ those of $(\mathcal{K}_1, \mathcal{C}_1)$; $\{A_1, \dots, A_n\}$ and $\{a_1, \dots, a_n\}$ those of $(\mathcal{M}_1, \mathcal{N}_1)$. Set $S = R, \Sigma = \Delta, \underline{m} = (1, \dots, 1) = \underline{n}$ in Definition 1.3.2 (ii)–(iii), then

$$\begin{aligned} \Upsilon &= \sum_{X \in \mathcal{T}} e_X * E_X \\ \Pi &= \sum_{j=1}^m v_j * V_j \\ \Theta &= \sum_{i=1}^n a_i * A_i \end{aligned} \tag{1.4-1}$$

are called the *formal products* of $(\mathcal{K}_0, \mathcal{C}_0)$, $(\mathcal{K}_1, \mathcal{C}_1)$ and $(\mathcal{M}_1, \mathcal{N}_1)$ respectively.

Lemma 1.4.1 Let δ be the differential in the bocs \mathfrak{B} . We have

$$\begin{aligned} (\sum_{i=1}^m v_i * V_i) (\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^m \mu_{11}(v_l) * V_l; \\ (\sum_{i=1}^n a_i * A_i) (\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^n \tau_{11}(a_l) * A_l; \\ (\sum_{j=1}^m v_j * V_j) (\sum_{i=1}^n a_i * A_i) &= \sum_{l=1}^n \iota_{11}(a_l) * A_l; \\ (\sum_{j=1}^m v_j * V_j) H - H (\sum_{j=1}^m v_j * V_j) &= \sum_{l=1}^n \partial_1(a_l) * A_l; \\ (\sum_{l=1}^n a_l * A_l) (\sum_{X \in \mathcal{T}} e_X * E_X) - (\sum_{X \in \mathcal{T}} e_X * E_X) (\sum_{l=1}^n a_l * A_l) &= \sum_{l=1}^n \delta(a_l) * A_l. \end{aligned}$$

Proof 1) The second equality is proved first, and the proofs of the first one and the third one are similar. By Lemma 1.3.3 (iii) for $S = R, p = q = 2$, the left side = $\sum_{l=1}^n (\sum_{i,j} \sigma_{ijl} \otimes_{R \otimes 3} (a_i \otimes_R v_i)) * A_l$ = the right side.

2) For the fourth equality, by Lemma 1.3.3 (i) the left side = $\sum_{l=1}^n (\sum_{j=1}^m \zeta_{lj} \otimes_{R \otimes 2} v_j) * A_l$ = the right side.

3) For the last one, by Lemma 1.3.3 (ii) for $p = 1, q = 2$, the left side = $\sum_{l=1}^n (a_l \otimes_R e_{Y_l} - e_{X_l} \otimes_R a_l) * A_l$ = the right side. \square

The matrix equation $(\Theta + H)(\Upsilon + \Pi) = (\Upsilon + \Pi)(\Theta + H)$, more precisely,

$$\begin{aligned} & (\sum_{i=1}^n a_i * A_i + H) (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j) \\ &= (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j) (\sum_{i=1}^n a_i * A_i + H) \end{aligned} \quad (1.4-2)$$

is called the *formal equation of the pair* $(\mathfrak{A}, \mathfrak{B})$ due to the following theorem.

Theorem 1.4.2 Let (p_l, q_l) be the leading position of A_l for $l = 1, \dots, n$. Then the (p_l, q_l) -entry of the formal equation is

$$\delta(a_l) = \iota_{11}(a_l) - \tau_{11}(a_l) + \partial_1(a_l).$$

Proof. According to Formula (1.4-2) and Lemma 1.4.1:

$$\begin{aligned} \sum_{l=1}^n \delta(a_l) * A_l &= \sum_{l=1}^n (a_l e_{t(a_l)} - e_{s(a_l)} a_l) * A_l \\ &= \sum_{j,i} (v_j * V_j) (a_i * A_i) - \sum_{i,j} (a_i * A_i) (v_j * V_j) + \sum_j ((v_j * V_j) H - H (v_j * V_j)) \\ &= \sum_{l=1}^n \iota_{11}(a_l) * A_l - \sum_{l=1}^n \tau_{11}(a_l) * A_l + \sum_{l=1}^n \partial_1(a_l) * A_l \\ &= \sum_{l=1}^n (\iota_{11}(a_l) - \tau_{11}(a_l) + \partial_1(a_l)) * A_l. \end{aligned}$$

The expression at the leading position (p_l, q_l) of the formal equation is obtained. \square

Moreover, the first formula of Lemma 1.4.1 gives:

$$\begin{aligned} & (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{i=1}^m v_i * V_i) (\sum_{X \in \mathcal{T}} e_X * E_X + \sum_{j=1}^m v_j * V_j) \\ &= \sum_{X \in \mathcal{T}} (e_X \otimes_R e_X) * E_X + \sum_{l=1}^m \mu(v_l) * V_l. \end{aligned} \quad (1.4-3)$$

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with an index set T and a vertex set \mathcal{T} . A size vector $\underline{m} = (m_1, \dots, m_t)$ over T , and a dimension vector $\underline{d} = (d_X \mid X \in \mathcal{T})$ over \mathcal{T} are said to be *associated*, if $m_i = d_X$ for $i \in X$.

Corollary 1.4.3 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair. Then the representation categories $R(\mathfrak{A})$ and $R(\mathfrak{B})$ are equivalent.

Proof Let $P \in R(\mathfrak{B})$ with dimension vector \underline{d} . Without loss of generality, the set $\{P(x) = W_X \mid X \in \mathcal{T}\}$ may be assumed to be a set of regular Weyr matrices. Then \bar{P} of size vector \underline{m} associated with \underline{d} in Definition 1.3.4 and P in Formula (1.2-5) are one-to-one corresponding; \bar{f}

in Definition 1.3.5 and f in Formula (1.2-6) are one-to-one corresponding. Moreover, $\bar{P}\bar{f} = \bar{f}\bar{Q}$ if and only if f satisfies Formula (1.2-7) by Theorem 1.4.2. \square

Thanks to Corollary 1.4.3, the representations and morphisms in both categories $R(\mathfrak{A})$ and $R(\mathfrak{B})$ can be denoted by P, f in a unified manner. Finally we define a special class of matrix bimodule problems to end the subsection. Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bimodule problem with R trivial. \mathfrak{A} is said to be *bipartite* provided that $\mathcal{T} = \mathcal{T}' \dot{\cup} \mathcal{T}''$; $R = R' \times R''$ and $\mathcal{K} = \mathcal{K}' \times \mathcal{K}''$ are direct products of algebras; and \mathcal{M} is a \mathcal{K}' - \mathcal{K}'' -bimodule.

Remark 1.4.4 Let Λ be a finite-dimensional basic k -algebra, $J = \text{rad}(\Lambda)$ be the Jacobson radical of Λ with the nilpotent index m , and the top $S = \Lambda/J$. Suppose $\{e_1, \dots, e_h\}$ is a complete set of orthogonal primitive idempotents of Λ . Taking the pre-images of k -bases of $e_i(J^i/J^{i+1})e_j$ under the canonical projections $J^i \rightarrow J^i/J^{i+1}$ in turn for $i = m, \dots, 1$, an ordered basis of J under the length order is obtained, see [CB1, 6.1] for details. Then we construct the left regular representation $\bar{\Lambda}$ of Λ under the k -basis $(a_n, \dots, a_2, a_1, e_1, \dots, e_h)$ of Λ , which leads to a bipartite matrix bimodule problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with

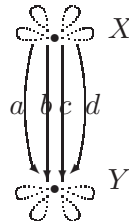
$$R = S \times S; \quad \mathcal{K}_0 \oplus \mathcal{K}_1 = \bar{\Lambda} \times \bar{\Lambda}; \quad \mathcal{M}_1 = \text{rad}(\bar{\Lambda}); \quad H = 0.$$

A simple calculation shows that the *row indices* of the leading positions of the base matrices in \mathcal{A} are *pairwise different*, and the *column index* of the leading position of $A \in \mathcal{A}_{XZ}$ equals $j_Z = \max\{j \in Z\}$ for any $X \in \mathcal{T}$, they are *concentrated*, and the j_Z -th column is said to be the *main column* over Z . Such a condition is denoted by RDCC for short. The condition may not be essential in the proof of the main theorem, but makes it easier and more intuitive.

Example 1.4.5 [D1, R1] Let $Q = a \circ \bullet \bullet \circ b$ be a quiver, $I = \langle a^2, ba - ab, ab^2, b^3 \rangle$ be an ideal of kQ , and $\Lambda = kQ/I$. Denote the residue classes of e, a, b in Λ still by e, a, b respectively. Moreover, set $c = b^2, d = ab$. Then an ordered k -basis $\{d, c, b, a, e\}$ of Λ gives a regular representation $\bar{\Lambda}$. A matrix bimodule problem \mathfrak{A} follows from Remark 1.4.4, we may denote by A, B, C, D the R - R -quasi-basis of \mathcal{M}_1 , and by a, b, c, d the R - R -dual basis of \mathcal{N}_1 . Then the associated boc \mathfrak{B} of \mathfrak{A} has a layer $L = (R; \omega; a, b, c, d; u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$. The formal equation of the pair $(\mathfrak{A}, \mathfrak{B})$ can be written as:

$$\begin{pmatrix} e & 0 & u_1 & u_2 & u_4 \\ & e & u_2 & 0 & u_3 \\ & & e & 0 & u_2 \\ & & & e & u_1 \\ & & & & e \end{pmatrix} \begin{pmatrix} 0 & 0 & a & b & d \\ & 0 & b & 0 & c \\ & & 0 & 0 & b \\ & & & 0 & a \\ & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & b & d \\ & 0 & b & 0 & c \\ & & 0 & 0 & b \\ & & & 0 & a \\ & & & & 0 \end{pmatrix} \begin{pmatrix} f & 0 & v_1 & v_2 & v_4 \\ & f & v_2 & 0 & v_3 \\ & & f & 0 & v_2 \\ & & & f & v_1 \\ & & & & f \end{pmatrix}$$

with $e = e_X, f = e_Y$ for simplicity. The differentials of the solid arrows of \mathfrak{B} can be read off according to Theorem 1.4.2:



$$\begin{cases} \delta(a) = 0, \\ \delta(b) = 0, \\ \delta(c) = u_2b - bv_2, \\ \delta(d) = u_1b + u_2a - bv_1 - av_2. \end{cases}$$

2 Reductions on matrix bimodule problems

In this section, the reduction theorem and eight reductions on matrix bimodule problems corresponding to those on bocses are stated and proved, thus the induced matrix bimodule

problems are constructed. Finally, a concept of defining systems of pairs is defined in order to help to construct the induced pairs in a sequence of reductions.

2.1 Admissible bimodules and induced matrix bimodule problems

In the subsection we prove the reduction theorem on matrix bimodule problems via admissible bimodules; then give the connection to the corresponding admissible functors and the reduction theorem on bocses. Before doing so, the following lemma is mentioned first.

Lemma 2.1.1 Let D be a commutative algebra, and Λ, Σ be commutative D -algebras. Suppose ${}_{\Lambda}\mathcal{G}$ and \mathcal{S}_{Σ} are finitely generated projective left Λ -module and right Σ -module respectively.

- (i) $\mathcal{G} \otimes_D \mathcal{S}$ is a projective $\Lambda \otimes_D \Sigma$ -module.
- (ii) There exists a $\Lambda \otimes_D \Sigma$ -module isomorphism:

$$\mathrm{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \otimes_D \mathrm{Hom}_{\Sigma}(\mathcal{S}, \Sigma) \cong \mathrm{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma).$$

Proof (i) Suppose ${}_{\Lambda}\mathcal{G}, \mathcal{S}_{\Sigma}$ are both free with the basis $\{u_1, \dots, u_m\}, \{v_1, \dots, v_n\}$ respectively. Choose a free $\Lambda \otimes_D \Sigma$ -module \mathcal{F} with basis $\{w_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. For $x = \sum_{i=1}^m \lambda_i u_i \in \mathcal{G}$ and $y = \sum_{j=1}^n v_j \sigma_j \in \mathcal{S}$, we define $f : \mathcal{G} \times \mathcal{S} \rightarrow \mathcal{F}, (x, y) = \sum_{i=1}^m \sum_{j=1}^n (\lambda_i \otimes \sigma_j) w_{ij}$. Then $xr = \sum_{i=1}^m (\lambda_i r) u_i$, and $ry = \sum_{j=1}^n v_j (r \sigma_j)$ for any $r \in D$, and hence $f(xr, y) = f(x, ry)$, $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ and $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$. Thus there exists a unique $\Lambda \otimes_D \Sigma$ -linear map $\tilde{f} : \mathcal{G} \otimes_D \mathcal{S} \rightarrow \mathcal{F}$ such that $\tilde{f}(x \otimes y) = f(x, y), \forall x \in \mathcal{G}, y \in \mathcal{S}$. In particular, $\tilde{f}(u_i \otimes v_j) = w_{ij}$. Thus $\{u_i \otimes v_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a $\Lambda \otimes_D \Sigma$ -basis of $\mathcal{G} \otimes_D \mathcal{S}$, and $\mathcal{G} \otimes_D \mathcal{S}$ is free. If both ${}_{\Lambda}\mathcal{G}, \mathcal{S}_{\Sigma}$ are projective, then there are some ${}_{\Lambda}\mathcal{G}', \mathcal{S}'_{\Sigma}$, such that both ${}_{\Lambda}\mathcal{G} \oplus \mathcal{S}_{\Sigma}, {}_{\Lambda}\mathcal{G}' \oplus \mathcal{S}'_{\Sigma}$ being free. The assertion follows.

(ii) It is stressed, that $\mathcal{G} \otimes_D \mathcal{S}$ and $\mathrm{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma)$ are projective $\Lambda \otimes_D \Sigma$ -modules by (i). Consider the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \times \mathrm{Hom}_{\Sigma}(\mathcal{S}, \Sigma) & \longrightarrow & \mathrm{Hom}_{\Lambda}(\mathcal{G}, \Lambda) \otimes_D \mathrm{Hom}_{\Sigma}(\mathcal{S}, \Sigma) \\ \psi \downarrow & \swarrow \tilde{\psi} & \\ \mathrm{Hom}_{\Lambda \otimes_D \Sigma}(\mathcal{G} \otimes_D \mathcal{S}, \Lambda \otimes_D \Sigma) & & \end{array}$$

Let $f \in \mathrm{Hom}_{\Lambda}(\mathcal{G}, \Lambda)$ and $g \in \mathrm{Hom}_{\Sigma}(\mathcal{S}, \Sigma)$. Since f and g are D -linear, there exists a $\Lambda \otimes_D \Sigma$ -linear map $\psi(f, g) : \mathcal{G} \otimes_D \mathcal{S} \rightarrow \Lambda \otimes_D \Sigma$, such that $(\psi(f, g))(x \otimes y) = f(x) \otimes g(y)$, for $(x, y) \in \mathcal{G} \times \mathcal{S}$. Now $\psi(fr, g) = \psi(f, rg)$ for $r \in D$, thus there exists a unique $(\Lambda \otimes_D \Sigma)$ -linear map $\tilde{\psi}$ given by $f \otimes g \mapsto \psi(f, g)$, which is clearly natural in both $\mathrm{Hom}_{\Lambda}(\mathcal{G}, \Lambda)$ and $\mathrm{Hom}_{\Sigma}(\mathcal{S}, \Sigma)$. $\tilde{\psi}$ is an isomorphism if ${}_{\Lambda}\mathcal{G}, \mathcal{S}_{\Sigma}$ are free, consequently, $\tilde{\psi}$ is an isomorphism if ${}_{\Lambda}\mathcal{G}, \mathcal{S}_{\Sigma}$ are projective. \square

Next we introduce a notion of admissible bimodules which is a module-theory version of admissible functor [CB1]. Some preliminaries are needed.

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with a minimal algebra R . Recall from Formula (1.2-4), that $\delta(a_i) = 0$ for the first h arrows $a_i, i = 1, \dots, h$, of \mathfrak{B} , if and only if $l(\mathcal{K}_1 \times \mathcal{M}_1), r(\mathcal{M}_1 \times \mathcal{K}_1), d(\mathcal{M}_1) \subseteq \mathcal{M}_1^{(h)}$. The algebra $\bar{R} = R[a_1, \dots, a_h]$ is said to be *pre-minimal*.

Let $\underline{d} = (n_X \mid X \in \mathcal{T})$ be a dimension vector over \mathcal{T} , R' a minimal algebra with the vertex set \mathcal{T}' and the algebra $\Delta' = \sum_{p=1}^{\infty} R'^{\otimes p}$, see Formula (1.1-3). Define an $R' \text{-} \bar{R}$ -bimodule L (or an \bar{R} -module over R') of dimension vector \underline{d} as follows: $L = \oplus_{X \in \mathcal{T}} L_X$, where $L_X = \oplus_{p=1}^{n_X} R'_{Z(X,p)}^*$ with $Z(X,p) \in \mathcal{T}'$, be an $R' \text{-} \bar{R}$ -bimodule. Let $L^* = \oplus_{X \in \mathcal{T}} L_X^*$ be the R' -dual module of L , where $L_X^* = \mathrm{Hom}_{R'}(L_X, R') = \oplus_{p=1}^{n_X} R'_{Z(X,p)}^*$. Clearly, L^* is an $\bar{R} \text{-} R'$ -bimodule.

Denote by $\mathbf{e}_{Z(X,p)}$ the $(1 \times n_X)$ -matrix with the p -th entry $1_{Z(X,p)}$ and others zero. Then the set $\{\mathbf{e}_{Z(X,p)} \mid 1 \leq p \leq n_X\}$ forms an R' -quasi-basis of L_X . Similarly, the set $\{\mathbf{f}_{Z(X,p)} = \mathbf{e}_{Z(X,p)}^* =$

$\{e_{Z(X,p)}^T \mid 1 \leq p \leq n_X\}$ of $(n_X \times 1)$ -matrices forms an R' -quasi-basis of L^* , where the superscript “ T ” stands for the transpose of matrix. Note that $f_{Z(Y,q)}(e_{Z(X,p)}) = e_{Z(X,p)} f_{Z(Y,q)} = 1_{Z(X,p)}$ for $(X,p) = (Y,q)$, or zero otherwise. And $f_{Z(X,p)} e_{Z(X,q)}$ is a matrix unit with the (p,q) -th entry $1_{Z(X,p)}$ for $Z(X,p) = Z(X,q)$, or a zero matrix otherwise. Define

$$\begin{aligned}\hat{E}_d &= \oplus_{X \in \mathcal{T}} (\hat{E}_d)_X, \quad (\hat{E}_d)_X = \oplus_{p=1}^{n_X} R'(f_{Z(X,p)} e_{Z(X,p)}) R' \subseteq \mathbb{D}_{n_X}(R'), \\ \hat{E}_u &= \oplus_{X \in \mathcal{T}} (\hat{E}_u)_X, \quad (\hat{E}_u)_X = \oplus_{1 \leq p < q \leq n_X} R'(f_{Z(X,p)} \otimes_k e_{Z(X,q)}) R' \subseteq \mathbb{N}_{n_X}(R' \otimes_k R'), \\ \hat{E}_l &= \oplus_{X \in \mathcal{T}} (\hat{E}_l)_X, \quad (\hat{E}_l)_X = \oplus_{1 \leq q < p \leq n_X} R'(f_{Z(X,p)} \otimes_k e_{Z(X,q)}) R' \subseteq \mathbb{N}_{n_X}(R' \otimes_k R')^T.\end{aligned}$$

In the following, $c : X \rightarrow Y$ is an arrow of \bar{R} , and the set $\mathcal{S} = \{x \mid X \in \mathcal{T}_1\} \cup \{a_1, \dots, a_h\}$.

$$\begin{aligned}\bar{E}_0 &= \{(B_X)_{X \in \mathcal{T}} \in \hat{E}_d \mid B_X L(c) = L(c) B_Y, \forall c \in \mathcal{S}\}, \\ \bar{E}_1 &= \{(B_X)_{X \in \mathcal{T}} \in \hat{E}_u \mid B_X L(c) = L(c) B_Y, \forall c \in \mathcal{S}\}, \\ \bar{E}_l &= \{(B_X)_{X \in \mathcal{T}} \in \hat{E}_l \mid B_X L(c) = L(c) B_Y, \forall c \in \mathcal{S}\}.\end{aligned}$$

Definition 2.1.2 With the notations as above. The R' - \bar{R} -bimodule L of dimension vector \underline{d} is said to be *admissible* provided that

- (a1) L is sincere over R' ;
- (a2) $\bar{E}_0 \simeq R'$ with an R' -quasi-basis:

$$\mathcal{F}_0 = \{F_Z = (F_{ZX})_{X \in \mathcal{T}} \mid Z \in \mathcal{T}'\}, \quad F_{ZX} = \sum_{Z(X,p)=Z} f_{Z(X,p)} e_{Z(X,p)};$$

- (a3) \bar{E}_1 is a quasi-free R' - R' -bimodule with a quasi-basis \mathcal{F}_1 for $i = 1, \dots, l$:

$$\mathcal{F}_1 = \{F_i = \sum_{p_{iu} < q_{iu}} \varepsilon_{X_i, p_{iu} q_{iu}} (f_{Z(X_i, p_{iu})} \otimes_k e_{Z(X_i, q_{iu})}) \mid \varepsilon_{X_i, p_{iu} q_{iu}} = 0 \text{ or } 1\},$$

where $\{(p_{iu}, q_{iu}) \mid \varepsilon_{X_i, p_{iu} q_{iu}} = 1\} \cap \{(p_{j\kappa}, q_{j\kappa}) \mid \varepsilon_{X_j, p_{j\kappa} q_{j\kappa}} = 1\} = \emptyset, \forall X_i = X_j$;

- (a4) $\bar{E}_l = \{0\}$.

The k -algebra $\bar{E} = \bar{E}_0 \oplus (\bar{\Delta}' \otimes_{R' \otimes 2} \bar{E}_1) \subseteq \Pi_{X \in \mathcal{T}} \mathbb{T}_{n_X}(\Delta')$ may be called a *pseudo endomorphism algebra* of L , which is finitely generated in index $(0, 1)$.

Let $M = \oplus_{X \in \mathcal{T}} M_X \subseteq \oplus_{X \in \mathcal{T}} \mathbb{N}_{n_X}(R' \otimes_k R')$ be an R' - R' -bimodule, where M_X has an R' - R' -quasi-basis $\{E_{(Xpq)} \mid p < q\}$, the matrix units of size n_X with the (p,q) -entry $1_{Z(X,p)} \otimes_k 1_{Z(X,q)}$ and others zero. There is an R' - R' -isomorphism $\kappa : \hat{E}_u \rightarrow M, f_{Z(X,p)} \otimes_k e_{Z(X,q)} \mapsto E_{(Xpq)}$. Furthermore, $M_X, \forall X \in \mathcal{T}$, possesses an \bar{R} - \bar{R} -bimodule structure as follows: if $b, c \in \Lambda$ with $t(b) = X = s(c)$, then $b \otimes_{\bar{R}} E_{Xpq} \otimes_{\bar{R}} c = L(b) E_{Xpq} L(c)$. It is clear that κ is also an \bar{R} - \bar{R} -bimodule isomorphism, and \hat{E}_u may be identified with M . Thus \bar{E}_1 can be viewed as a submodule of M .

Write the quasi-free R' - R' -bimodule $L^* \otimes_k L = \oplus_{(X,Y) \in \mathcal{T} \times \mathcal{T}} L_X^* \otimes_k L_Y$. An induced matrix bimodule \mathfrak{A}' of \mathfrak{A} based on an admissible bimodule is described below.

Construction 2.1.3 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem. Suppose $\bar{R}, R', L, \underline{d}, \bar{E}$ are given as in Definition 2.1.2. Then there exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ in the following sense.

(i) Let $\underline{n} = (n_1, \dots, n_t)$ be the size vector associated with \underline{d} stated before Corollary 1.4.3. Define a set of integers $T' = \{1, \dots, t'\}$ with $t' = \sum_{i \in T} n_i$. Then the set of vertices of R' is the partition \mathcal{T}' of T' , and the matrices in $\mathcal{K}', \mathcal{M}'$ and H' are of size $\underline{n} \times \underline{n}$ partitioned under \mathcal{T} .

(ii) The k -algebra \mathcal{K}' is given as follows. First, let $\mathcal{F}'_0 = \{E'_Z = \sum_{X \in \mathcal{T}} F_{Z,X} * E_X \in \mathbb{D}_{t'}(R') \mid Z \in \mathcal{T}'\}$ by Definition 1.3.2 (ii) for $S = R', p = 1$, and \mathcal{K}'_0 be an algebra generated by \mathcal{F}'_0 over R' . \mathcal{F}'_0 is a quasi-basis of \mathcal{K}'_0 via the isomorphism $\bar{E}_0 \xrightarrow{\nu_0} \mathcal{K}'_0, F_Z \mapsto E'_Z$, and $\mathcal{K}'_0 \simeq R'$.

Second, let $\mathcal{F}'_1 = \{F'_i = F_i * E_{X_i} \mid i = 1, \dots, l\}$ by 1.3.2 (ii) for $S = R', p = 2$, and \mathcal{K}'_{10} be an $R'-R'$ -bimodule generated by \mathcal{F}'_1 , where \mathcal{F}'_1 is a quasi-basis via the isomorphism $\bar{E}_1 \xrightarrow{\nu} \mathcal{K}'_{10}, F_i \mapsto F'_i$. There exists a natural order on \mathcal{F}'_1 according to the leading positions of matrices. Let $\mathcal{K}'_{11} = (L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{K}_1 \subseteq \mathbb{N}'(R' \otimes_k R')$ be an $R'-R'$ -bimodule with a quasi-basis:

$$\mathcal{U}' = \{(f_{Z_{(X'_j, p)}} \otimes_k e_{Z_{(Y'_j, q)}}) * V_j \mid V_j \in \mathcal{V}_{X'_j Y'_j}, 1 \leq p \leq n_{X'_j}, 1 \leq q \leq n_{Y'_j}, 1 \leq j \leq m\}$$

given by Definition 1.3.2 (iii) for $S = R', p = 2$. Finally, set $\mathcal{K}'_1 = \mathcal{K}'_{10} \oplus \mathcal{K}'_{11}$ with a quasi-basis $\mathcal{V}' = \mathcal{F}'_1 \cup \mathcal{U}'$.

(iii) Let $\mathcal{M}'_1 = (L^* \otimes_k L) \otimes_{R^{\otimes 2}} \mathcal{M}_1$ be an $R'-R'$ -bimodule with a normalized quasi-basis

$$\mathcal{A}' = \{(f_{Z_{(X_i, p)}} \otimes_k e_{Z_{(Y_i, q)}}) * A_i \mid A_i \in \mathcal{A}_{X_i Y_i}, 1 \leq p \leq n_{X_i}, 1 \leq q \leq n_{Y_i}, h < i \leq n\}$$

given by Definition 1.3.2 (iii) for $S = R', p = 2$.

(iv) Let $H' = \sum_{X \in \mathcal{T}} H_X(L_X(x)) + \sum_{i=1}^h L(a_i) * A_i$ be a matrix over R' , where $L_X(x) = \bar{W}_X$, $H_X(\bar{W}_X)$ is defined in 1.3.2 (i); and $*$ is given by 1.3.2 (iii) for $S = R', p = 1$.

(v) The product $m'_{11} : (\mathcal{K}'_{10} \oplus \mathcal{K}'_{11}) \times (\mathcal{K}'_{10} \oplus \mathcal{K}'_{11}) \rightarrow \mathcal{K}'_2$ is given by

$$\begin{aligned} & ((f_{Z_{(X'_i, p_1)}} \otimes_k e_{Z_{(Y'_i, q_1)}}) * V_i) ((f_{Z_{(X'_j, p_2)}} \otimes_k e_{Z_{(Y'_j, q_2)}}) * V_j) \\ &= \sum_l \left(((f_{Z_{(X'_i, p_1)}} \otimes_k e_{Z_{(Y'_i, q_1)}}) (f_{Z_{(X'_j, p_2)}} \otimes_k e_{Z_{(Y'_j, q_2)}})) \otimes_{R^{\otimes 3}} \gamma_{ijl} \right) * V_l, \\ & (F_i * E_{X_i}) ((f_{X'_l, p} \otimes_k e_{Z_{(Y'_l, q)}}) * V_l) = ((F_i (f_{Z_{(X'_l, p)}} \otimes_k e_{Z_{(Y'_l, q)}})) * (E_{X_i} V_l), \\ & ((f_{Z_{(X'_l, p)}} \otimes_k e_{Z_{(Y'_l, q)}}) * V_l) (F_i * E_{X_i}) = ((f_{Z_{(X'_l, p)}} \otimes_k e_{Z_{(Y'_l, q)}}) F_i) * (V_l E_{X_i}), \\ & (F_i * E_{X_i}) (F_j * E_{X_j}) = (F_i F_j) * (E_{X_i} E_{X_j}) \end{aligned}$$

according to 1.3.3 (iii) and (ii) for $p = 2 = q$. The left module action $l'_{11} : (\mathcal{K}'_{10} \oplus \mathcal{K}'_{11}) \times \mathcal{M}'_1 \rightarrow \mathcal{M}'_2$ is similar to the first and second formulae above, the right one $r'_{11} : \mathcal{M}'_1 \times (\mathcal{K}'_{10} \oplus \mathcal{K}'_{11}) \rightarrow \mathcal{M}'_2$ to the first and third ones. Finally, the derivation $d_1 = (d_{10}, d_{11})$ with $d'_{10} : \mathcal{K}'_{10} \rightarrow \{0\}$ and

$$d'_{11} : \mathcal{K}'_{11} \rightarrow \mathcal{M}'_1, (f_{Z_{(X'_j, p)}} \otimes_k e_{Z_{(Y'_j, q)}}) * V_j \mapsto \sum_l (\zeta_{jl} \otimes_{R^{\otimes 2}} (f_{Z_{(X'_j, p)}} \otimes_k e_{Z_{(Y'_j, q)}})) * A_l. \quad \square$$

Admissible bimodules can be transferred to admissible functors as follows. A minimal algebra R can be viewed as a minimal category $A' = \prod_{X \in \mathcal{T}_0} \text{mod } R_X \times \prod_{X \in \mathcal{T}_1} P(R_X)$, [CB1, 2.1] by the one-to-one correspondence between the vertex set of R and the set of indecomposable objects of A' , the two sets may be identified for the sake of convenience. Then \bar{R} determines a pre-minimal category A'' by adding some morphisms $a_i : X_i \mapsto Y_i, i = 1, \dots, h$, into A' . A similar transfer holds from R' to B' . Thus L can be viewed as a functor $\theta' : A' \rightarrow B'$, where

$$\theta'(X) = \bigoplus_{p=1}^{n_X} Z_{(X, p)}, \quad \theta'(c) = L(c), \quad \forall c \in \Lambda. \quad (2.1-1)$$

We stress, that the opposite construction is usually impossible. Throughout the paper, the right module structure and upper triangular matrix are mainly used, which is opposite to the left module and lower triangular matrix used in [CB1].

Proposition 2.1.4 The functor θ' is admissible in the sense of [CB1, 4.3 Definition].

Proof 1) (A1) is clear; (a1) implies (A2); the finite set $\Lambda = \{Z_{(X, p)} \mid 1 \leq p \leq n_X, X \in \mathcal{T}\}$ has a partial order: $Z_{(X, p)} \prec Z_{(X, q)}$ if $p < q$ in (A3); (A4) follows from $e_{Z_{(X, p)}} f_{Z_{(Y, q)}} = 1_{Z_{X, p}}$ for $(X, p) = (Y, q)$, or 0 otherwise; (A5) from $\sum f_{Z_{(X, p)}} e_{Z_{(X, p)}} = 1_{L_X}$.

2) Let $\bar{E}_0^* = \text{Hom}_{R'}(\bar{E}_0, R')$, then $\bar{E}_0^* \simeq \text{Hom}_{R'}(R', R') \simeq R'$ by (a2). Since $(\hat{E}_d)_X \simeq \bigoplus_{p=1}^{n_X} R'(f_{Z_{(X, p)}} e_{Z_{(X, p)}}) \simeq \bigoplus_{p=1}^{n_X} R'(f_{Z_{(X, p)}} \otimes_{R'} R' e_{Z_{(X, p)}}) \simeq \bigoplus_{p=1}^{n_X} R'(f_{Z_{(X, p)}} \otimes_{R'} e_{Z_{(X, p)}}) R'$, Lemma 2.1.1 (ii) shows

$$\begin{aligned} & \text{Hom}_{R'}(L^* \otimes_{R'} L, R') \simeq \text{Hom}_{R' \otimes_{R'} R'}(L^* \otimes_{R'} L, R' \otimes_{R'} R') \\ & \simeq \text{Hom}_{R'}(L^*, R') \otimes_{R'} \text{Hom}_{R'}(L, R') \simeq L \otimes_{R'} L^*. \end{aligned}$$

We claim, that $\bar{E}_0^* = \sum_{X \in \mathcal{T}, p} R'(\mathbf{e}_{Z(X,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(X,p)})R'$. In fact, $\hat{E}_d^* \subseteq L \otimes_{R'} L^*$, and \hat{E}_d^* has an \bar{R} - \bar{R} -bimodule structure given by $b(\mathbf{e}_{Z(X,p)} \otimes_{R'} \mathbf{f}_{Z(X,p)})c = (\mathbf{e}_{Z(X,p)} L(b)) \otimes_{R'} (L(c) \mathbf{f}_{Z(X,p)})$, $\forall b, c \in \Lambda$ with $s(b) = X = t(c)$. The \bar{R} - \bar{R} -structures on \hat{E}_d and \hat{E}_d^* ensure that $\bar{E}_0^* = \{(B_X^*)_{X \in \mathcal{T}} \in \hat{E}_d^* \mid cB_X^* = B_Y^*c, \forall c \in S\}$ by [J, Proposition 3.4, 3.5] as claimed.

3) Let $\bar{E}_1^* = \text{Hom}_{R' \otimes 2}(\bar{E}_1, R'^{\otimes 2})$. Lemma 2.1.1 (ii) shows:

$$\text{Hom}_{R' \otimes_k R'}(L^* \otimes_k L, R' \otimes_k R') \simeq \text{Hom}_{R'}(L^*, R') \otimes_k \text{Hom}_{R'}(L, R') \simeq L \otimes_k L^*. \quad (2.1-2)$$

By (a3) and a similar argument as in 2), $\bar{E}_1^* \simeq \sum_{X \in \mathcal{T}, p < q} R'(\mathbf{e}_{Z(X,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(X,q)})R'$.

4) Combining 2), 3) and noting $\bar{E}_l^* = \sum_{X \in \mathcal{T}, p > q} R'(\mathbf{e}_{Z(X,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(X,q)})R' = \{0\}$ by (a4), $\bar{E}_0^* \oplus \bar{E}_1^* = L \otimes_{\bar{R}} L^*$. Recall from Formula (2.1-1) that $L \otimes_{\bar{R}} L^*$ corresponds to $B' \otimes_{A'} B'$. There is an exact sequence $0 \rightarrow \bar{E}_1^* \rightarrow \bar{E}_1^* \oplus \bar{E}_0^* \xrightarrow{p} \bar{E}_0^* \rightarrow 0$, $p(\mathbf{e}_{Z(X,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(Y,q)}) = \mathbf{e}_{Z(X,p)} \mathbf{f}_{Z(Y,q)}$, which corresponds to the map $B' \otimes_{A'} B' \rightarrow B'$ in [CB1, 4.3 (A3)]. Thus the kernel J' of the map corresponds to \bar{E}_1^* , and J' is projective from \bar{E}^* being so. (A3) follows.

5) (A6) concerns the bimodule action on $\bar{E}_0^* \simeq R'$. Since $\mathbf{e}_{Z(X,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(X,q)} = 0$ for $p > q$ in \bar{E}_l^* , (A7) follows. \square

Proposition 2.1.5 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, and let \mathfrak{A}' be given by Construction 2.1.3. Then the associated boc \mathfrak{B}' of \mathfrak{A}' is the induced boc of \mathfrak{B} given by [CB1, 4.5 Proposition].

Proof Denote by $\mathfrak{C}' = (R', \mathcal{C}', \mathcal{N}', \partial')$ the associated bi-comodule problem of \mathfrak{A}' .

1) $\mathcal{C}'_0 = \text{Hom}_{R'}(\mathcal{K}'_0, R')$. The isomorphism $\bar{E}_0^* = \text{Hom}_{R'}(\bar{E}_0, R') \xrightarrow{\nu_0^*} \text{Hom}_{R'}(\mathcal{K}'_0, R') = \mathcal{C}'_0$ with ν_0^* being the R' -dual map of ν_0 in 2.1.3 (ii) gives the R' -quasi-basis $\mathcal{F}_0'^* = \{e'_Z = \nu_0^*(F_Z^*) \mid Z \in \mathcal{T}'\}$ of \mathcal{C}'_0 , see the proof 2) of Proposition 2.1.4. And $\mathcal{F}_0'^*$ is R' -dual to \mathcal{F}_0' of \mathcal{K}'_0 .

2) $\mathcal{C}'_1 = \text{Hom}_{R' \otimes 2}(\mathcal{K}'_{10} \oplus \mathcal{K}'_{11}, R'^{\otimes 2}) = \mathcal{C}'_{10} \oplus \mathcal{C}'_{11}$. Since

$$\bar{E}_1^* = \text{Hom}_{R' \otimes 2}(\bar{E}_1, R'^{\otimes 2}) \xrightarrow{\nu_1^*} \text{Hom}_{R' \otimes 2}(\mathcal{K}'_{10}, R'^{\otimes 2}) = \mathcal{C}'_{10}$$

is an isomorphism with ν_1^* being the $R'^{\otimes 2}$ -dual map of ν_1 in 2.1.3 (ii), $\mathcal{F}_1'^* = \{F_i'^* = \nu_1^*(F_i^*) \mid i = 1, \dots, l\}$ forms an R' - R' -basis of \mathcal{C}'_{10} . And $\mathcal{F}_1'^*$ inherits a linear order from \mathcal{F}_1' . Since R' is an R - R -bimodule via the isomorphism $R' \simeq \mathcal{K}'_0$, by Lemma 2.1.1 (ii) and Formula (2.1-2):

$$\begin{aligned} \mathcal{C}'_{11} &= \text{Hom}_{R' \otimes 2}(\mathcal{K}'_{11}, R'^{\otimes 2}) = \text{Hom}_{R' \otimes 2}((L^* \otimes_k L) \otimes_{R \otimes 2} \mathcal{K}_1, R'^{\otimes 2}) \\ &\simeq \text{Hom}_{R' \otimes 2 \otimes_{R \otimes 2} R \otimes 2}((L^* \otimes_k L) \otimes_{R \otimes 2} \mathcal{K}_1, R'^{\otimes 2} \otimes_{R \otimes 2} R \otimes 2) \\ &\simeq \text{Hom}_{R' \otimes 2}(L^* \otimes_k L, R'^{\otimes 2}) \otimes_{R \otimes 2} \text{Hom}_{R \otimes 2}(\mathcal{K}_1, R \otimes 2) \simeq (L \otimes_k L^*) \otimes_{R \otimes 2} \mathcal{C}_1. \end{aligned}$$

Write $(\mathbf{e}_{Z(X'_j,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(Y'_j,q)}) \otimes_{R \otimes 2} v_j = \mathbf{e}_{Z(X'_j,p)} \otimes_{\bar{R}} v_j \otimes_{\bar{R}} \mathbf{f}_{Z(Y'_j,q)} = v_{j p q}$. $\mathcal{U}'^* = \{v_{j p q} \mid 1 \leq p \leq n_{s(v_j)}, 1 \leq q \leq n_{t(v_j)}; 1 \leq j \leq m\}$ is an R' - R' -quasi-basis of \mathcal{C}'_{11} dual to \mathcal{U}' of \mathcal{K}_{11} given in Construction 2.1.3 (ii). The R' - R' -quasi basis of \mathcal{C}'_1 is $\mathcal{V}'^* = \mathcal{F}_1'^* \cup \mathcal{U}'^*$ dual to \mathcal{V}' of \mathcal{K}'_1 .

3) $\mathcal{N}'_1 = \text{Hom}_{R' \otimes 2}(\mathcal{M}'_1, R'^{\otimes 2}) \simeq (L \otimes_k L^*) \otimes_{R \otimes 2} \mathcal{M}_1$ can be proved in a similar manner as that for \mathcal{C}'_{11} . Write $(\mathbf{e}_{Z(X_i,p)} \otimes_{\bar{R}} \mathbf{f}_{Z(Y_i,q)}) \otimes_{R \otimes 2} a_i = \mathbf{e}_{Z(X_i,p)} \otimes_{\bar{R}} a_i \otimes_{\bar{R}} \mathbf{f}_{Z(Y_i,q)} = a_{i p q}$. $\mathcal{A}'^* = \{a_{i p q} \mid 1 \leq p \leq n_{s(a_i)}, 1 \leq q \leq n_{t(a_i)}; h < i \leq n\}$ is an R' - R' -quasi basis of \mathcal{N}'_1 dual to \mathcal{A}' of \mathcal{M}'_1 .

4) Formula (1.4-1) shows the formal products of $(\mathcal{K}'_0, \mathcal{C}'_0)$, $(\mathcal{K}'_1, \mathcal{C}'_1)$, $(\mathcal{M}'_1, \mathcal{N}'_1)$ respectively:

$$\begin{aligned} \Upsilon' &= \sum_{Z \in \mathcal{T}'} e_{Z'} * E'_Z; \\ \Pi' &= \sum_{j,p,q} v_{j p q} * ((\mathbf{f}_{Z(X'_j,p)} \otimes_{\bar{R}} \mathbf{e}_{Z(Y'_j,q)}) * V_j) + \sum_{i=1}^l F_i'^* * F'_i \\ &= \sum_{j=1}^m (v_{j p q})_{n_{X'_j} \times n_{Y'_j}} * V_j + \sum_{i=1}^l F_i'^* F'_i = \Pi'_1 + \Pi'_0; \\ \Theta' &= \sum_{i,p,q} a_{i p q} * ((\mathbf{f}_{Z(X_i,p)} \otimes_{\bar{R}} \mathbf{e}_{Z(Y_i,q)}) * A_i) = \sum_{i=1}^n (a_{i p q})_{n_{X_i} \times n_{Y_i}} * A_i \end{aligned}$$

Exhibit the formal equation $\Theta'\Upsilon' - \Upsilon'\Theta' = (\Pi'_1\Theta' - \Theta'\Pi'_1 + \Pi'_1H' - H'\Pi'_1) + \Pi'_0\Theta' - \Theta'\Pi'_0$:

$$\begin{aligned} & \sum_l (a_{lpq} \otimes_{R'} e_{t(a_{lpq})}) * A_l - \sum_l (e_{s(a_{lpq})} \otimes_{R'} a_{lpq}) * A_l \\ &= \sum_{l,(i,j)} \left(((v_{ipq})(a_{j pq})) \otimes_{R^{\otimes 3}} \eta_{ijl} \right) * A_l - \sum_{l,(i,j)} \left(((a_{ipq})(v_{j pq})) \otimes_{R^{\otimes 3}} \sigma_{ijl} \right) * A_l \\ &+ \sum_{l,j} (\zeta_{jl} \otimes_{R^{\otimes 2}} (v_{j pq})) * A_l + \sum_{l,i} (F_i'^*(a_{lpq})) * (F_i' A_l) - \sum_{l,i} ((a_{lpq}) F_i'^*) * (A_l F_i'), \end{aligned}$$

where $(v_{ipq})(a_{j pq}) = (\sum_h v_{iph} \otimes_{R'} a_{jhq})$, and other matrix products are similar. Thus Theorem 1.4.2 shows the differential δ' in \mathfrak{B}' :

$$\begin{aligned} \delta'(a_{lpq}) &= e_{Z(X_l, p)} \otimes_R \delta(a_l) \otimes_R f_{Z_{Y_l, q}} \\ &+ \sum_{p'} \nu_1^*(e_{Z(X_l, p)} \otimes_{\bar{R}} f_{Z(Y_l, p')}) \otimes_{R'} a_{lp'q} - \sum_{q'} a_{lpq'} \otimes_{R'} \nu_1^*(e_{Z(X_l, q')} \otimes_{\bar{R}} f_{Z(Y_l, q)}). \end{aligned}$$

This coincides with the formula in [CB1, 4.5 Proposition], i.e. \mathfrak{B}' is the induced bocs of \mathfrak{B} . \square

2.2 Eight reductions

In this subsection seven reductions of matrix bimodule problems based on Definition 2.1.2 and Construction 2.1.3 are introduced, where the last two do not occur in any references on bocses. And finally, a regularization is presented as the eighth reduction.

Proposition 2.2.1 (Localization) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$ and $R'_X = k[x, \phi(x)^{-1}c(x)^{-1}]$ a finitely generated localization of R_X . Define two algebras $\bar{R} = R$, $R' = R'_X \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$, and an R' - \bar{R} -bimodule $L = R'$. Then L is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ of \mathfrak{A} and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced bocs \mathfrak{B}' of \mathfrak{B} given by localization [CB1, 4.8] is the associated bocs of \mathfrak{A}' .

Proposition 2.2.2 (Loop mutation) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the first arrow $a_1 : X \mapsto X$, such that $\delta(a_1) = 0, X \in \mathcal{T}_0$. Define a pre-minimal algebra $\bar{R} = R[a_1]$, a minimal algebra $R' = R'_X \times \prod_{Y \in \mathcal{T} \setminus \{X\}} R_Y$ with $R'_X = k[x]$, and an R' - \bar{R} -bimodule $L = R'$. Then L is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', d')$ of \mathfrak{A} , and an equivalent functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced bocs \mathfrak{B}' of \mathfrak{B} given by the functor $\theta' : A' \rightarrow B'$, with $\theta'(Y) = Y, \forall Y \in \mathcal{T}$, $\theta'(a_1) = x$, is the associated bocs of \mathfrak{A}' by Proposition 2.1.5.

Proposition 2.2.3 (Deletion) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair, $\mathcal{T}' \subset \mathcal{T}$. Define two algebras $\bar{R} = R$, $R' = \prod_{X \in \mathcal{T}'} R_X$, and an R' - \bar{R} -bimodule $L = R'$. Then L is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ of \mathfrak{A} , and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced bocs \mathfrak{B}' of \mathfrak{B} obtained by deletion of $\mathcal{T} \setminus \mathcal{T}'$ [CB1, 4.6] is the associated bocs of \mathfrak{A}' .

Let the algebra $R_X = k[x, \phi(x)^{-1}]$, $r \in \mathbb{Z}^+$, and $\lambda_1, \dots, \lambda_s \in k$ with $\phi(\lambda_i) \neq 0$. Write $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$. Define a minimal algebra S and an S - R_X -bimodule K :

$$\begin{aligned} S &= \left(\prod_{i=1}^s \prod_{j=1}^r k1_{Z_{ij}} \right) \times k[z, \phi(z)^{-1}g(z)^{-1}]; \\ K &= \left(\oplus_{i=1}^s \oplus_{l=1}^r \oplus_{j=1}^l k1_{Z_{ijl}} \right) \oplus k[z, \phi(z)^{-1}g(z)^{-1}], \quad Z_{ijl} = Z_{ij}; \\ K(x) &= \bar{W} : K \rightarrow K, \quad \bar{W} \simeq \oplus_{i=1}^s \oplus_{j=1}^r J_j(\lambda_j)1_{Z_{ij}} \oplus (z1_{Z_0}), \end{aligned} \tag{2.2-1}$$

where \bar{W} is a Weyl matrix over S . Let $n = \frac{1}{2}sr(r+1) + 1$. Denote by $\{(i, j, l) \mid 1 \leq j \leq r, 1 \leq l \leq j, 1 \leq i \leq s\} \cup \{n\}$ the index set of the direct summands of K . There is a partition given by $Z_{ij} = \{(i, j, l) \mid l = 1, \dots, j\}$, $Z = \{n\}$. An order on the set is defined as

$$(i, j, l) \prec (i', j', l') \iff i < i'; \text{ or } i = i', l < l'; \text{ or } i = i', l = l', j > j',$$

and $n \prec (i, j, l)$. Let $\mathbf{e}_{(ijl)}$ be a $1 \times n$, (resp. $\mathbf{f}_{(ijl)}$ an $n \times 1$) matrix with $1_{Z_{ij}}$ at the (i, j, l) -th component and 0 at others. Then K has an S - S -quasi-basis $\{\mathbf{e}_{(ijl)}, \mathbf{e}_n \mid 1 \leq j \leq r, 1 \leq l \leq j, 1 \leq i \leq s\}$, and $K^* = \text{Hom}_S(K, S)$ has $\{\mathbf{f}_{(ijl)}, \mathbf{f}_n \mid 1 \leq j \leq r, 1 \leq l \leq j, 1 \leq i \leq s\}$. The S -quasi-free-module \bar{E}_0 , and the S - S -quasi-free bimodule \bar{E}_1 have the quasi-basis respectively:

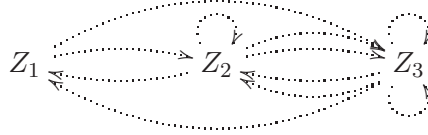
$$\begin{aligned} \{F_{ij} = \sum_{l=1}^j \mathbf{f}_{(ijl)} \otimes_S \mathbf{e}_{(ijl)}; F_n = \mathbf{f}_n \otimes_S \mathbf{e}_n \mid 1 \leq j \leq r, 1 \leq i \leq s\}, \\ F_{ijj'l} = \sum_{h=1}^{j'-l+1} \mathbf{f}_{(ij'h)} \otimes_k \mathbf{e}_{(ij', l+h-1)}, \quad l = \begin{cases} 1, \dots, j', & \text{if } j > j'; \\ 2, \dots, j', & \text{if } j = j'; \\ j, \dots, j', & \text{if } j < j'. \end{cases} \end{aligned} \quad (2.2-2)$$

Proposition 2.2.4 (Unraveling) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $R_X = k[x, \phi(x)^{-1}]$. Define two algebras $\bar{R} = R$, $R' = S \times \prod_{Z \in \mathcal{T} \setminus \{X\}} R_Z$, and an R' - \bar{R} -bimodule $L = K \oplus (\oplus_{Z \in \mathcal{T} \setminus \{X\}} R_Z)$ with S and K given by Formula (2.2-1). Then L is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ and a fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by unraveling [CB1, 4.7] is the associated boc of \mathfrak{A}' .

The picture below shows $\mathbf{e}_{(ij1)} \otimes_{\bar{R}} \mathbf{f}_{(ij'l)}$ in \bar{E}_1^* as dotted arrows for $s = 1, r = 3$:



Let an algebra \bar{R}_{XY} , a minimal algebra S and an S - \bar{R}_{XY} -module K be defined as follows:

$$\begin{aligned} \bar{R}_{XY} : X \xrightarrow{a_1} Y; \quad S = \prod_{i=1}^3 S_{Z_i}, \quad S_{Z_i} = k1_{Z_i}, i = 1, 2, 3; \\ K_X = k1_{Z_2} \oplus k1_{Z_1}, \quad K_Y = k1_{Z_3} \oplus k1_{Z_2}, \quad K(a_1) = \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix} : K_X \rightarrow K_Y. \end{aligned} \quad (2.2-3)$$

Let $Z_{(X,1)} = Z_2 = Z_{(Y,2)}$, and $Z_{(X,2)} = Z_1, Z_{(Y,1)} = Z_3$, then $\{\mathbf{e}_{Z_{(X,1)}}, \mathbf{e}_{Z_{(X,2)}}\}$ is an S -quasi-basis of K_X , and $\{\mathbf{f}_{Z_{(X,1)}}, \mathbf{f}_{Z_{(X,2)}}\}$ is that of $K_X^* = \text{Hom}_S(K_X, S)$. There is a similar observation on K_Y . The S -quasi-free module \bar{E}_0 , and the S - S -quasi-free bimodule \bar{E}_1 have the quasi-basis respectively:

$$\begin{aligned} F_{Z_1} = \mathbf{f}_{Z_{(X,2)}} \otimes_S \mathbf{e}_{Z_{(X,2)}}, \quad F_{Z_3} = \mathbf{f}_{Z_{(Y,1)}} \otimes_S \mathbf{e}_{Z_{(Y,1)}}, \\ F_{Z_2} = (\mathbf{f}_{Z_{(X,1)}} \otimes_S \mathbf{e}_{Z_{(X,1)}}; \mathbf{f}_{Z_{(Y,2)}} \otimes_S \mathbf{e}_{Z_{(Y,2)}}); \\ F_{Z_2 Z_1} = \mathbf{f}_{Z_{(X,1)}} \otimes_k \mathbf{e}_{Z_{(X,2)}}, \quad F_{Z_3 Z_2} = \mathbf{f}_{Z_{(Y,1)}} \otimes_k \mathbf{e}_{Z_{(Y,2)}}. \end{aligned} \quad (2.2-4)$$

Proposition 2.2.5 (Edge reduction) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the first arrow $a_1 : X \mapsto Y$, such that $X, Y \in \mathcal{T}_0, \delta(a_1) = 0$. Define a pre-minimal algebra $\bar{R} = R[a_1]$, a minimal algebra $R' = S \times \prod_{Z \in \mathcal{T} \setminus \{X, Y\}} R_Z$, and an R' - \bar{R} -bimodule $L = K \oplus (\oplus_{Z \in \mathcal{T} \setminus \{X, Y\}} R_Z)$ with S and K defined in Formula (2.2-3). Then L is admissible.

(i) There exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$, and an equivalent functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by edge reduction [CB1, 4.9] is the associated boc of \mathfrak{A}' .

Proposition 2.2.6 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the first arrow $a_1 : X \mapsto Y$, such that $X, Y \in \mathcal{T}_0, \delta(a_1) = 0$. Set two algebras $\bar{R} = R[a_1]$, $R' = R$, and an R' - \bar{R} -bimodule $L = K \oplus (\oplus_{U \in \mathcal{T} \setminus \{X, Y\}} R_U)$ with $K : R_X \xrightarrow{(0)} R_Y$. Then L is admissible

(i) There are an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', d')$ with $\mathcal{K}' = \mathcal{K}$, $\mathcal{M}' = \mathcal{M}^{(1)}$, $H' = H$, and an induced fully faithful functors $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of representations P with $P(a_1) = 0$ is equivalent to $R(\mathfrak{A}')$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by the admissible functor $\theta' : A' \rightarrow B'$ with $\theta'(U) = U, \forall U \in \mathcal{T}$ and $\theta'(a_1) = 0$ is the associated boc of \mathfrak{A}' .

Let $R_{XY} = R_X \times R_Y$ be a minimal algebra with $R_X = k[x, \phi(x)^{-1}]1_X$, $R_Y = k1_Y$. Define an algebra $\bar{R}_{XY} = R_{XY}[a_1]$ with $a_1 : X \rightarrow Y$, an algebra $S = k[z, \phi(z)^{-1}]$, and an S - \bar{R}_{XY} -bimodule K with $K_X = S$, $K_Y = S$, $K(x) = (z)$, $K(a_1) = (1_Z)$. Then there are a S -module $\bar{E}_0 = SF_Z$ with F_Z defined below, and a S - S -bimodule $\bar{E}_1 = 0$.

$$\begin{aligned} \bar{R}_{XY} : x \bigcirc X \xrightarrow{a_1} Y; \quad S : z \bigcirc Z; \quad K : z \bigcirc S \xrightarrow{(1)} S; \\ F_Z = (f_{z_X} \otimes_S e_{z_X}, f_{z_Y} \otimes_S e_{z_Y}) = (1_Z, 1_Z). \end{aligned} \quad (2.2-5)$$

Proposition 2.2.7 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with the first arrow $a_1 : X \mapsto Y$ and $\delta(a_1) = 0$. Define a pre-minimal algebra $\bar{R} = R[a_1]$, a minimal algebra $R' = S \times \prod_{U \in \mathcal{T} \setminus \{X, Y\}} R_U$, and an R' - \bar{R} -bimodule $L = K \oplus (\oplus_{U \in \mathcal{T} \setminus \{X, Y\}} R_U)$, where S and K are given by Formula (2.2-5). Then L is admissible.

(i) There are an induced matrix bimodule problem \mathfrak{A}' , and an induced fully faithful functors $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. The subcategory of $R(\mathfrak{A})$ consisting of representations P with $P(a_1)$ invertible is equivalent to $R(\mathfrak{A}')$.

(ii) The induced boc \mathfrak{B}' given by the admissible functor $\theta' : A' \rightarrow B'$ with $\theta'(X) = Z$, $\theta'(Y) = Z$; $\theta'(x) = z$, $\theta'(a_1) = (1)$, is the associated boc of \mathfrak{A}' .

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, d)$ be a matrix bimodule problem, $\mathfrak{C} = (R, \mathcal{C}, \mathcal{N}, \partial)$ be the associated bi-comodule problem, and \mathfrak{B} the boc of \mathfrak{C} . Then

$$\begin{aligned} d(V_1) = A_1 + \sum_{l \geq 1} \zeta_{1l} A_l \text{ and } d(V_j) \in \mathcal{M}_1^{(1)} \text{ for } j \geq 2 \text{ in } \mathfrak{A} \\ \iff \partial(a_1) = v_1 \text{ in } \mathfrak{C} \iff \delta(a_1) = v_1 \text{ in } \mathfrak{B}. \end{aligned} \quad (2.2-6)$$

In fact, since $\partial(a_1) = \sum_{j=1}^m \zeta_{j1} v_j$, we have $\partial(a_1) = v_1$, if and only if $\zeta_{11} = 1_{s(a_1)} \otimes_k 1_{t(a_1)}$ and $\zeta_{j1} = 0$ for all $j \geq 2$, if and only if $d(V_1) = A_1 + \sum_{l \geq 1} \zeta_{1l} A_l$ and $d(V_j) \in \mathcal{M}^{(2)}$ for all $j \geq 2$, since $d(V_j) = \zeta_{j1} A_1 + \sum_{i \geq 1} \zeta_{ji} A_i$. On the other hand, noting $\iota_1(a_1) = 0$ and $\tau_1(a_1) = 0$ by triangularity of \mathfrak{C} , thus $\delta(a_1) = v_1$ in \mathfrak{B} , if and only if $\partial(a_1) = v_1$ in \mathfrak{C} by the definition of $\delta_1 : \Gamma \rightarrow \bar{\Omega}$ given below Lemma 1.2.5.

Remark Let $\mathfrak{A}, \mathfrak{C}$ be given as above with $\partial(a_1) = v_1$. Then

(i) $\mathcal{K}^{(1)} = \mathcal{K}_0 \oplus (\oplus_{j=2}^m \bar{\Delta} \otimes_{R \otimes 2} V_j)$ is a sub-algebra of \mathcal{K} , and $\mathcal{M}^{(1)} = \oplus_{i=2}^n \bar{\Delta} \otimes_{R \otimes 2} a_i$ is a $\mathcal{K}^{(1)}$ - $\mathcal{K}^{(1)}$ -sub-bimodule;

(ii) $\mathcal{C}^{(1)} = \bar{\Delta} \otimes_{R \otimes 2} v_1$ is a coideal of \mathcal{C} , and $\mathcal{C}^{[1]} = \mathcal{C}/\mathcal{C}^{(1)}$ is a quotient coalgebra; $\mathcal{N}^{(1)} = \bar{\Delta} \otimes_{R \otimes 2} a_1$ is a \mathcal{C} - \mathcal{C} -sub-bi-comodule, and $\mathcal{N}^{[1]} = \mathcal{N}/\mathcal{N}^{(1)}$ is a $\mathcal{C}^{[1]}$ - $\mathcal{C}^{[1]}$ -quotient bi-comodule.

Proof (i) By the triangularity (1.2-2), $d(V_i V_j) = d(V_i) V_j + V_i d(V_j) \in \mathcal{M}^{(1)}$ and $\forall V_i, V_j \in \mathcal{V}$:

$$d(V_i V_j) = d(\sum_{l=1}^m \gamma_{ijl} \otimes_{R \otimes 2} V_l) = \sum_{l=1}^m \gamma_{ijl} \otimes_{R \otimes 2} d(V_l) = \sum_{l,p} (\gamma_{ijl} \otimes_{R \otimes 2} \zeta_{lp}) \otimes_{R \otimes 2} A_p.$$

The coefficient of A_1 is $\sum_l (\gamma_{ijl} \otimes_{R \otimes 2} \zeta_{l1}) = 0$, where $\zeta_{11} = 1$, $\zeta_{l1} = 0$ for $l > 1$ by hypothesis, so that $\gamma_{ij1} = 0$ for all $1 \leq i, j \leq m$. Therefore $V_i V_j = \sum_{l \geq 1} \gamma_{ijl} \otimes_{R \otimes 2} V_l \in \mathcal{K}^{(1)}$ and hence $\mathcal{K}^{(1)}$ is a subalgebra of \mathcal{K} . $\mathcal{M}^{(1)}$ is a $\mathcal{K}^{(1)}$ - $\mathcal{K}^{(1)}$ -bimodule deduced from the triangularity (1.2-2) easily.

(ii) Since $\mu(v_1) = \mu(\partial(a_1)) = (\partial \otimes \mathbb{1})(\iota(a_1)) + (\mathbb{1} \otimes \partial)(\tau(a_1)) \stackrel{(1.2-3)}{=} (\partial \otimes \mathbb{1})(e_{s(a_1)} \otimes_R a_1) + (\mathbb{1} \otimes \partial)(a_1 \otimes_R e_{t(a_1)}) = 0$, $\mathcal{C}^{(1)}$ is a coideal of \mathcal{C} . $\mathcal{N}^{(1)}$ is a \mathcal{C} - \mathcal{C} -sub-bi-comodule deduced from (1.2-3) easily. \square

Proposition 2.2.8 (Regularization) Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with $\delta(a_1) = v_1$.

(i) There is an induced matrix bimodule problem $\mathfrak{A}' = (R, \mathcal{K}^{(1)}, \mathcal{M}^{(1)}, H)$ of \mathfrak{A} , and an equivalent functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$.

(ii) The induced boc \mathfrak{B}' of \mathfrak{B} given by regularization [CB1, 4.2] is the associated boc of \mathfrak{A}' . \square

Proof (i) \mathfrak{A}' is a matrix bimodule problem by Remark (i) above. Note that $R' = R, \mathcal{T}' = \mathcal{T}$. Taken any $P \in R(\mathfrak{A})$ of size vector \underline{m} , let $f = \sum_{X \in \mathcal{T}} I_{m_X} * E_X + P(a_1) * V_1$, then $P' = f^{-1} P f \in R(\mathfrak{A}')$. Therefore, ϑ is an equivalent functor.

(ii) $\mathfrak{C}' = (R, \mathcal{C}^{[1]}, \mathcal{N}^{[1]}, \bar{\partial})$ with $\bar{\partial}$ induced from ∂ is the associated bi-comodule problem of \mathfrak{A}' by Remark (ii) above. Thus the associated boc \mathfrak{B}' of \mathfrak{A}' is given by regularization from \mathfrak{B} . \square

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ in \mathfrak{B} . Suppose the first arrow $a_1 : X \mapsto Y$ with $\delta(a_1) = \sum_{j=1}^m f_j(x, y) v_j \neq 0$. In order to obtain $\delta(a_1) = h(x, y) v'_1$, we make the following base change:

$$(v'_1, \dots, v'_m) = (v_1, \dots, v_m) F(x, y), \quad (2.2-7)$$

where $F(x, y) \in \text{Im}(R \otimes_k R)$ is invertible. When $X \in \mathcal{T}_0$ or $Y \in \mathcal{T}_0$, R is preserved; but when $X, Y \in \mathcal{T}_1$, some localization $R'_X = R_X[c(x)^{-1}]$ (resp. $R'_Y = R_Y[c(y)^{-1}]$) is needed [CB1, §5]. Consequently, we have a base change of \mathcal{K}_1 dually given by

$$(V'_1, \dots, V'_m) = (V_1, \dots, V_m) F(x, y)^{-T}. \quad (2.2-8)$$

Finally, a simple fact according to all the reductions defined above is mentioned to end the subsection. Let us start from a matrix bimodule problem $\mathfrak{A}^0 = (R^0, \mathcal{K}^0, \mathcal{M}^0, H = 0)$ with \mathcal{T}^0 trivial, and after a series of reductions, an induce matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ is obtained. Then for any $X \in \mathcal{T}'$, $H'_X = (h_{ij}(x))$, any entry $h_{ij}(x) = a_{ij} + b_{ij}x \in k[x]$.

2.3 Canonical forms

In this subsection, a canonical form (cf.[S]) for each representation of a matrix bimodule problem is calculated; and a notion of reduction blocks is defined.

Convention 2.3.1 Suppose \mathfrak{A} is a matrix bimodule problem, \mathfrak{A}' an induced matrix bimodule problem and $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ an induced functor. Let \underline{m}' be a size vector over \mathcal{T}' of \mathfrak{A}' . A size vector $\underline{m} = (m_1, m_2, \dots, m_t)$ over \mathcal{T} of \mathfrak{A} based on \underline{m}' is defined:

- (i) for regularization, loop mutation, localization, and Proposition 2.2.6, set $\underline{m} = \underline{m}'$;
- (ii) for deletion, set $m_i = m'_i$ if $i \in X, X \in \mathcal{T}'$, and $m_i = 0$ if $i \in X, X \in \mathcal{T} \setminus \mathcal{T}'$;
- (iii) for edge reduction, set $m_i = m'_i$ if $i \in Z, Z \neq X, Y$, $m_i = m'_{Z_1} + m'_{Z_2}$ if $i \in X$, and $m_i = m'_{Z_2} + m'_{Z_3}$ if $i \in Y$; For proposition 2.2.7, set $m_X = m'_Z, m_Y = m'_Z$;
- (iv) for unraveling, set $m_i = m'_i$ if $i \notin X$, and $m_i = \sum_{i=1}^s \sum_{j=1}^r j m'_{Z_{ij}} + m'_{Z_0}$ if $i \in X$.

Then \underline{m} is said to be the *size vector determined by \underline{m}'* , and is denoted by $\vartheta(\underline{m}')$.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem with \mathcal{T} being trivial, and \underline{m} be a size vector. For the sake of simplicity, we write

$$H_{\underline{m}}(k) = \sum_{X \in \mathcal{T}} H_X(I_{m_X}), \quad H(k) = \sum_{X \in \mathcal{T}} H_X(1). \quad (2.3-1)$$

Let P be a representation of size vector \underline{m} in $R(\mathfrak{A})$. Then Definition 1.3.4 shows:

$$P = H_{\underline{m}}(k) + \sum_{i=1}^n P(a_i) * A_i. \quad (2.3-2)$$

Let $\mathcal{T}' = \{i \mid m_i \neq 0\}$, and the induced bimodule problem \mathfrak{A}' be given by a deletion of $\mathcal{T} \setminus \mathcal{T}'$. Then the size vector $\underline{m}' = (m_i \mid m_i \neq 0)$ is sincere over \mathcal{T}' . It may be assumed that \underline{m} is

sincere over \mathcal{T} in the sequel. We will find an induced matrix bimodule problem \mathfrak{A}' given by minimal steps of reductions, and an object $P' \in R(\mathfrak{A}')$ of sincere size vector \underline{m}' over \mathcal{T}' , such that $\vartheta(P') \simeq P$ under the induced functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. Let \mathfrak{B} be the associated boc of \mathfrak{A} with the first arrow $a_1 : X \rightarrow Y$. There are three possibilities.

(i) If $\delta(a_1) = v_1$, we proceed with a regularization, and obtain an induced matrix bimodule problem \mathfrak{A}' . Set

$$B = (\emptyset)_{m_X \times m_Y}, \quad G = (\emptyset)_{1 \times 1}, \quad (2.3-3)$$

where \emptyset indicates a distinguished zero entry or block. Suppose P' is given in the proof (i) of Proposition 2.2.8 with $\mathcal{T}' = \mathcal{T}$, $\underline{m}' = \underline{m}$. Then $P'(a_1) = B$ and $\vartheta(P') \simeq P$ in $R(\mathfrak{A})$.

(ii) If $\delta(a_1) = 0$ and $X = Y$, suppose $P(a_1) \simeq J = \bigoplus_{i=1}^s (\bigoplus_{j=1}^r J_j(\lambda_i)^{e_{ij}})$, $e_{ij} \geq 0$, a Jordan form over k with the maximal size r of the Jordan blocks. We first proceed with a loop mutation $a_1 \mapsto (x)$, then with an unraveling for the polynomial $g(x) = (x - \lambda_1) \cdots (x - \lambda_s)$ and the integer r , thus an induced matrix bimodule problem \mathfrak{A}_1 of \mathfrak{A} is obtained. Let $f_X \in \text{IM}_{m_X}(k)$ be invertible, such that

$$B = f_X^{-1} P(a_1) f_X = W; \quad G = \bar{W}, \quad (2.3-4)$$

where W is a Weyr matrix over k , \bar{W} is that over R' similar to $\bigoplus_{i=1}^s \bigoplus_{e_{ij} > 0} J_j(\lambda_i) 1_{Z_{ij}}$. Deleting a set of vertices $\{Z_0\} \cup \{Z_{ij} \mid e_{ij} = 0\}$ from \mathfrak{A}_1 , an induced problem \mathfrak{A}' of \mathfrak{A} is obtained. Let $\underline{m}' = (m'_i)_{i \in \mathcal{T}'}$ be a size vector over \mathcal{T}' with $m'_Z = m_Z$ for $Z \in \mathcal{T} \setminus \{X\}$, and $m'_{Z_{ij}} = j e_{ij}$, then \underline{m}' is sincere. Let $f = f_X * E_X + \sum_{Z \in \mathcal{T} \setminus \{X\}} I_{m_Z} * E_Z$, and $P' = f^{-1} P f$ with the size vector \underline{m}' in $R(\mathfrak{A}')$. Then $P'(a_1) = B$ and $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$.

(iii) If $\delta(a_1) = 0$ and $X \neq Y$, we proceed with an edge reduction for \mathfrak{A} and obtain an induced problem \mathfrak{A}_1 with the vertex set \mathcal{T}_1 . If $\text{rank}(P(a_1)) = r$, let $f_X \in \text{IM}_{m_X}(k)$, $f_Y \in \text{IM}_{m_Y}(k)$ be invertible, such that

$$B = f_X^{-1} P(a_1) f_Y = \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}_{m_X \times m_Y}. \quad (2.3-5)$$

Set $G = \textcircled{1}(0)$, $\textcircled{2}(1_{Z_2})$, $\textcircled{3}(0 \ 1_{Z_2})$, $\textcircled{4} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, or $\textcircled{5} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

where the five cases of G are obtained by deleting a subset $\hat{\mathcal{T}} \subset \mathcal{T}_1$ from \mathfrak{A}_1 : $\textcircled{1} \hat{\mathcal{T}} = \{Z_2\}$ for $r = 0$; now suppose $r > 0$, $\textcircled{2} \hat{\mathcal{T}} = \{Z_1, Z_3\}$ for $m_X = r = m_Y$; $\textcircled{3} \hat{\mathcal{T}} = \{Z_1\}$ for $m_X = r, m_Y > r$; $\textcircled{4} \hat{\mathcal{T}} = \{Z_3\}$ for $m_X > r, m_Y = r$; $\textcircled{5} \hat{\mathcal{T}} = \emptyset$ for $m_X, m_Y > r$. An induced matrix bimodule problem \mathfrak{A}' of \mathfrak{A} given by $a_1 \mapsto G$ is obtained. Let $\underline{m}' = (m'_i)_{i \in \mathcal{T}'}$ be a size vector over \mathcal{T}' , with $m'_Z = m_Z$ for $Z \in \mathcal{T} \setminus \{X, Y\}$, $m'_{Z_1} = m_X - r$, $m'_{Z_2} = r$ and $m'_{Z_3} = m_Y - r$, thus \underline{m}' is sincere over \mathcal{T}' . Let $f = f_X * E_X + f_Y * E_Y + \sum_{Z \in \mathcal{T} \setminus \{X, Y\}} I_{m_Z} * E_Z$, and $P' = f^{-1} P f$ with the size vector \underline{m}' in $R(\mathfrak{A}')$. Then $P'(a_1) = B$ and $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$.

Lemma 2.3.2 (cf.[S]) Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem with \mathcal{T} trivial, and let P be given in Formula (2.3-2). Then there exists an induced matrix bimodule problem $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ given by one of the following three reductions:

- (i) Regularization,
- (ii) Loop reduction: a loop mutation, then a unraveling, followed by a deletion.
- (iii) Edge reduction: first an edge reduction, followed by a deletion,

There is a representation P' of sincere size vector \underline{m}' over \mathcal{T}' in $R(\mathfrak{A}')$, such that $P \simeq \vartheta(P')$ in $R(\mathfrak{A})$ under the fully faithful functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. According to Formulae (2.3-3)–(2.3-5):

$$P' = H_{\underline{m}'}(k) + B * A_1 + \sum_{i=1}^{n'} P(a'_i) * A'_i.$$

The procedure may be said that *the reduction is given by* $a_1 \mapsto G$ in one of Formulae (2.3-3)–(2.3-5), and G is called the *reduction block from* \mathfrak{A} *to* \mathfrak{A}' .

Applying Lemma 2.3.2 repeatedly, the following theorem is obtained by induction.

Theorem 2.3.3 (cf.[S]) Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bimodule problem with \mathcal{T} trivial. Let $P \in R(\mathfrak{A})$ be a representation of sincere size vector \underline{m} . Then there exists a unique sequence of matrix bimodule problems:

$$\mathfrak{A} = \mathfrak{A}^0, \quad \mathfrak{A}^1, \quad \dots, \quad \mathfrak{A}^i, \quad \mathfrak{A}^{i+1}, \quad \dots, \quad \mathfrak{A}^s \quad (*)$$

$$(G^1, \quad \dots, \quad G^i, \quad G^{i+1}, \quad \dots, \quad G^s)$$

where \mathfrak{A}^{i+1} is obtained from \mathfrak{A}^i by $a_1^i \mapsto G^{i+1}$ defined by one of Formula (2.3-3)–(2.3-5). There is also a unique sequence of representations:

$$P^0 = P, \quad P^1, \quad \dots, \quad P^i, \quad P^{i+1}, \quad \dots, \quad P^s \quad (**)$$

$$(B^1, \quad \dots, \quad B^i, \quad B^{i+1}, \quad \dots, \quad B^s)$$

where $P^i(a_1^i) = B^i$ is defined by one of Formulae (2.3-3)–(2.3-5). Let $\vartheta^{i,i+1} : R(\mathfrak{A}^{i+1}) \rightarrow R(\mathfrak{A}^i)$ be the induced functor. There is a representation $P^{i+1} \in R(\mathfrak{A}^{i+1})$ of sincere size vector \underline{m}^{i+1} with $\vartheta^{i,i+1}(P^{i+1}) \simeq P^i$ for $0 \leq i < s$.

Write for $i < j$ the composition of induced functors $\vartheta^{ij} = \vartheta^{i,i+1} \dots \vartheta^{j-1,j} : R(\mathfrak{A}^j) \rightarrow R(\mathfrak{A}^i)$. Denote by A_1^i the first quasi-basis matrix of \mathcal{M}_1^i in \mathfrak{A}^i , then $P^{i+1} = H_{\underline{m}^i}^i(k) + B^{i+1} * A_1^i + \sum_{j=2}^{n^{i+1}} M^{i+1}(a_j^{i+1}) * A_j^{i+1}$. Using the formula inductively:

$$\vartheta^{0s}(H_{\underline{m}^s}^s(k)) = \sum_{i=0}^{s-1} B^{i+1} * A_1^i \in R(\mathfrak{A}). \quad (2.3-6)$$

In particular, if \mathfrak{A}^s is minimal, then $P^s = H_{\underline{m}^s}^s(k)$. In this case, the matrix $\vartheta^{0s}(P^s)$ is called the *canonical form* of P , and denoted by P^∞ . The entry “1” appearing in B^{i+1} of P^∞ , which is not an eigenvalue when B^{i+1} being a Weyr matrix, is called a *link* of P^∞ . And denote by $l(P^\infty)$ the number of the links in P^∞ .

Corollary 2.3.4 [S, XZ] The canonical form of any representation P over a matrix bimodule problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ with R trivial is uniquely determined. Moreover,

- (i) for any $P, Q \in R(\mathfrak{A})$, $P \simeq Q$ if and only if P and Q have the same canonical form;
- (ii) P is indecomposable if and only if $l(P^\infty) = \dim(P) - 1$.

Corollary 2.3.5 Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ be a matrix bimodule problem with \mathcal{T} trivial, let $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ be an induced matrix bimodule problem obtained by a series of reductions with an induced functor $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$. If \mathcal{T}' is trivial and $P = \vartheta(H'(k))$ with a sincere size vector \underline{m} over \mathcal{T} in $R(\mathfrak{A})$, then there is a unique reduction sequence $(*)$ performed for P by Theorem 2.3.3.

We conclude this subsection with a definition of some reduction block $G_s^{i+1}(R^s)$ (or G_s^{i+1} for short) over R^s . Under the hypothesis of Corollary 2.3.5, set $\mathfrak{A}' = \mathfrak{A}^s$ in the sequence $(*)$. Let $\underline{m}^s = (1, \dots, 1)$ and $\underline{m}^i = \vartheta^{is}(\underline{m}^s)$. Write $s(a_1^i) = X^i$ and $t(a_1^i) = Y^i$. A matrix $G_s^{i+1} \in \mathbb{M}_{m_{X^i}^i \times m_{Y^i}^i}^i(R^s)$ is determined by

- (i) $G_s^{i+1}(k) = G_s^{i+1} \otimes 1 \in \mathbb{M}_{m_{X^i}^i \times m_{Y^i}^i}^i(R^s) \otimes_{R^s} k \simeq \mathbb{M}_{m_{X^i}^i \times m_{Y^i}^i}^i(k)$ is equal to B^{i+1} given in Theorem 2.3.3;
- (ii) write the matrix $G_s^{i+1} * A_1^i = (g_{pq}) \in \mathbb{M}_{ts}(R^s)$, if $g_{pq} \neq 0$, and $p \in X^s$ (or equivalently, $q \in X^s$), then $g_{pq} \in R_{X^s}^s$.

And G_s^{i+1} for $i = 0, \dots, s-1$ are said to be the *reduction blocks* of H^s . Furthermore,

$$H^s = \sum_{i=0}^{s-1} G_s^{i+1} * A_1^i \quad \text{and} \quad \vartheta^{0s}(H^s(k)) = \sum_{i=0}^{s-1} G_i^{s+1}(k) * A_1^i. \quad (2.3-7)$$

For the sake of convenience, a links of $\vartheta^{0s}(H^s(k))$ is also said to be a link of H^s . Thus \mathfrak{A}^s is local if and only if $l(H^s) = \dim(\vartheta^{0s}(H^s(k))) - 1$.

2.4 Defining systems

We introduce a concept of defining systems in this subsection. There exist two sorts of systems used in different situations in order to construct induced matrix bimodule problems in a reduction sequence.

Let $B = (b_{ij})_{t \times t}$ and $C = (c_{ij})_{t \times t}$ be two $t \times t$ matrices over k . Given $1 \leq p, q \leq t$, the notation $B \equiv_{\prec(p,q)} C$ (resp. $B \equiv_{(p,q)} C$, $B \equiv_{\preceq(p,q)} C$) means that $b_{ij} = c_{ij}$ for any $(i, j) \prec (p, q)$ (resp. $(i, j) = (p, q)$, $(i, j) \preceq (p, q)$). One can define the similar notations for partitioned matrices.

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with \mathcal{T} trivial, $H = 0$, the R - R -quasi-basis $\mathcal{V} = \{V_1, \dots, V_m\}$ of \mathcal{K}_1 and $\mathcal{A} = \{A_1, \dots, A_n\}$ of \mathcal{M}_1 . Denote by (p_j, q_j) the leading position of A_j for $j = 1, \dots, n$. Suppose there exists a sequence of reductions in the sense of Lemma 2.3.2:

$$(\mathfrak{A}, \mathfrak{B}) = (\mathfrak{A}^0, \mathfrak{B}^0), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \dots, (\mathfrak{A}^r, \mathfrak{B}^r), \dots, (\mathfrak{A}^s, \mathfrak{B}^s). \quad (2.4-1)$$

For each $0 \leq i \leq s$ in the sequence, a matrix equation is defined by

$$\begin{aligned} \mathbb{E}^i : \quad & \Phi_{\underline{m}^i} H^i(k) \equiv_{\prec(p^i, q^i)} H^i(k) \Phi_{\underline{m}^i} \\ \text{with} \quad & \Phi_{\underline{m}^i} = \sum_{X \in \mathcal{T}} Z_X * E_X + \sum_{j=1}^m Z_j * V_j, \end{aligned} \quad (2.4-2)$$

where (p^i, q^i) is the leading position of A_1^i of \mathcal{M}_1^i in the i -th pair, $\underline{m}^i = \vartheta^{0i}(1, \dots, 1)$ is the size vector of $\vartheta^{0i}(H^i(k)) \in R(\mathfrak{A})$ over \mathcal{T} , $Z_X = (z_{pq}^X)_{m_X \times m_X}$, $\forall X \in \mathcal{T}$, and $Z_j = (z_{pq}^j)_{m_s(v_j) \times m_t(v_j)}$ for all quasi-base matrices V_j of \mathcal{K}_1 in \mathfrak{A} , z_{pq}^X, z_{pq}^j are pairwise different variables over k . $\Phi_{\underline{m}^i}$ is called a *variable matrix*. The system of linear equations in \mathbb{E}^i , which consists of equations locating in the (p_j, q_j) -th block for $j = 1, \dots, n$, is said to be a *defining system of \mathcal{K}^i* , and is denoted still by \mathbb{E}^i .

Theorem 2.4.1 The solution space of the defining system \mathbb{E}^i in Formula (2.4-2) is the k -vector space spanned by the quasi-basis of $\mathcal{K}_0^i \oplus \mathcal{K}_1^i$ for $0 \leq i \leq s$ in the sequence (2.4-1).

Proof Since $H^0 = H = 0$, our theorem holds true for $i = 0$. Suppose the theorem is true for the defining system \mathbb{E}^i , now consider the defining system \mathbb{E}^{i+1} .

In the case of Regularization, $\underline{m}^{i+1} = \underline{m}^i$, the k -vector space spanned by the quasi-basis of $\mathcal{K}_0^{i+1} \oplus \mathcal{K}_1^{i+1}$ is just the solution space of the equations of \mathbb{E}^i and the equation $\Phi_{\underline{m}^i} H^i(k) \equiv_{(p^i, q^i)} H^i(k) \Phi_{\underline{m}^i}$, which form the equation system $\mathbb{E}^{i+1} : \Phi_{\underline{m}^{i+1}} H^{i+1}(k) \equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1}(k) \Phi_{\underline{m}^{i+1}}$.

In the case of Loop or Edge reduction, set $a_1 : X \rightarrow Y$ with $X = Y$ for loop reduction; denote by $\underline{n} = \vartheta^{i, i+1}(1, \dots, 1)$, the size vector of $\vartheta^{i, i+1}(H^{i+1}(k))$ over \mathcal{T}^i . Then the size vector $\underline{m}^{i+1} = \vartheta^{0i}(\underline{n})$ over \mathcal{T} . Since $G^{i+1} = L(a_1)$, write $G^{i+1}(k) = L(a_1)(k)$. The k -vector space spanned by the quasi-basis of $\mathcal{K}_0^{i+1} \oplus \mathcal{K}_1^{i+1}$ is the solution space of the matrix equations partitioned under \mathcal{T}^i :

$$\begin{cases} (\Phi_{\underline{m}^i})_{\underline{n}} H_{\underline{n}}^i(k) \equiv_{\preceq(p^i, q^i)} H_{\underline{n}}^i(k) (\Phi_{\underline{m}^i})_{\underline{n}}, \\ (\Phi_{\underline{m}^i})_{\underline{n}, X} L(a_1)(k) = L(a_1)(k) (\Phi_{\underline{m}^i})_{\underline{n}, Y}, \end{cases}$$

where if $\Phi_{\underline{m}^i} = (x_{pq})$, then $(\Phi_{\underline{m}^i})_{\underline{n}} = (X_{pq})$ with $X_{pq} = (x_{pq, \alpha\beta})_{n_p \times n_q}$; since $\delta(a_1) = 0$, the (p^i, q^i) -th equation of \mathbb{E}^i is a linear combination of previous equations, “ $\preceq(p^i, q^i)$ ” can be used in the first formula; in the second one $(\Phi_{\underline{m}^i})_{\underline{n}, X}$ stands for the (j, j) -th block of $(\Phi_{\underline{m}^i})_{\underline{n}}$ with $j \in X$. Since $(\Phi_{\underline{m}^i})_{\underline{n}} = \Phi_{\underline{m}^{i+1}}$ and $H^{i+1}(k) = H_{\underline{n}}^i(k) + L(a_1)(k) * A_1^i$, the equation system above is just $\mathbb{E}^{i+1} : \Phi_{\underline{m}^{i+1}} H^{i+1}(k) \equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1}(k) \Phi_{\underline{m}^{i+1}}$, where (p^{i+1}, q^{i+1}) is the index followed by the biggest index over \mathcal{T}^{i+1} of the (p^i, q^i) -block partitioned under \mathcal{T}^i . Our theorem is proved by induction. \square

For an example, see 2.4.5 (iv) below. The theorem implies the following fact obviously.

Corollary 2.4.2 $\delta(a_1^i) = 0$ in \mathfrak{B}^i if and only if the (p^i, q^i) -equation of \mathbb{E}^i , is a linear combination of the equations of \mathbb{E}^i , namely, the equations locating before (p^i, q^i) .

Next, we give a deformed system based on the defining system. Fix some $0 < r < s$ in the sequence (2.4-1). Suppose $\mathfrak{A}^r = (R^r, \mathcal{K}^r, \mathcal{M}^r, H^r)$, and $\{V_1^r, \dots, V_{m^r}^r\}$ is a normalized quasi-basis of \mathcal{K}_1^r . If $i \geq r$, set a size vector over $\underline{m}^r = \vartheta^{ri}(1, \dots, 1)$ over \mathcal{T}^r , where $(1, \dots, 1)$ is a size vector over \mathcal{T}^i . Let $Z_{Y^r}^i$ be variable matrices of size $m_{Y^r}^{ri}, \forall Y^r \in \mathcal{T}^r$, and Z_j^{ri} be those of size $m_{s(v_j^r)}^{ri} \times m_{t(v_j^r)}^{ri}, 1 \leq j \leq m^r$, then set a variable matrix $\Psi_{\underline{m}^r} = \sum_{Y^r \in \mathcal{T}^r} Z_{Y^r}^{ri} * E_{Y^r}^r + \sum_{j=1}^{m^r} Z_j^{ri} * V_j^r$. Let $H^i = H_1^i + H_2^i$ with $H_1^i = \sum_{j=1}^{r-1} G_i^{j+1} * A_1^j, H_2^i = \sum_{j=r}^{i-1} G_i^{j+1} * A_1^j$. Then the matrix equation $\Psi_{\underline{m}^r} H^i(k) \equiv_{\prec(p^i, q^i)} H^i(k) \Psi_{\underline{m}^r}$ can be rewritten as

$$\begin{aligned} \mathbb{F}^{ri} : \Psi_{\underline{m}^r} H_2^i(k) &\equiv_{\prec(p^i, q^i)} \Psi_{\underline{m}^r}^0 + H_2^i(k) \Psi_{\underline{m}^r}, \\ \Psi_{\underline{m}^r}^0 &= H_1^i(k) \Psi_{\underline{m}^r} - \Psi_{\underline{m}^r} H_1^i(k). \end{aligned} \quad (2.4-3)$$

Corollary 2.4.3 The equation system \mathbb{F}^{ri} is equivalent to \mathbb{E}^i . And $\delta(a_1^i) = 0$ in \mathfrak{B}^i if and only if the (p^i, q^i) -th equation \mathbb{F}^i is a linear combination of equations of \mathbb{F}^i , namely, the equations locating before (p^i, q^i) .

The above theorem and corollaries will be used in Subsections 5.2–5.4 to calculate the differentials of bocses given by some bordered matrices. Sometimes, it is difficult to determine the dotted arrows in the induced boc after some reductions. Instead, we may consider a system of equations on “dotted elements” (see the definition below), and give explicitly the linear relations on those elements, which will be used in Subsections 4.1 and 4.3–4.5.

Theorem 2.4.4 For each $0 \leq i \leq s$ in the sequence (2.4-1), there exists a system of equations $\bar{\mathbb{E}}^i$ over $R^i \otimes_k R^i$, whose general solution can be expressed as the formal product $\Pi^i = \sum_j v_j^i * V_j^i$, namely

- (i) the $R^i \otimes_k R^i$ -quasi-basis $\{V_j^i\}_j$ of \mathcal{K}_1^i forms a basic system of solutions of $\bar{\mathbb{E}}^i$;
- (ii) the $R^i \otimes_k R^i$ -quasi-basis $\{v_j^i\}_j$ of \mathcal{C}_1^i forms a set of free variables.

Proof For $i = 0$, let $\bar{\Phi}_{\underline{m}^0} = \sum_{j=1}^m v_j * V_j = \Pi$ and $\bar{\mathbb{E}}^0 : \bar{\Phi}_{\underline{m}^0} H^0 \equiv_{\prec(p, q)} H^0 \bar{\Phi}_{\underline{m}^0}$ be a matrix equation with (p, q) being the leading position of A_1 in $\mathcal{M}_1, H^0 = 0$. Then Π is a general solution of $\bar{\mathbb{E}}^0$.

Suppose a system $\bar{\mathbb{E}}^i$ of the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ satisfying condition (i)–(ii) has been obtained:

$$\bar{\mathbb{E}}^i : \bar{\Phi}_{\underline{m}^i} H^i \equiv_{\prec(p^i, q^i)} H^i \bar{\Phi}_{\underline{m}^i}, \quad (2.4-4)$$

where $\bar{\Phi}^i = (u_{pq})$ is strictly upper triangular with u_{pq} being a k -linear combination of some variables over $R_X^i \times R_Y^i$ for $p \in X, q \in Y, X, Y \in \mathcal{T}^i$. We now construct a system $\bar{\mathbb{E}}^{i+1}$. For the sake of convenience, $\bar{\mathbb{E}}_{\prec(p^i, q^i)}^i$ is used for the equation system $\bar{\mathbb{E}}^i$, and $\bar{\mathbb{E}}_{(p^i, q^i)}^i$ stands for the (p^i, q^i) -th equation of $\bar{\mathbb{E}}^i$.

1) If $\bar{\mathbb{E}}_{(p^i, q^i)}^i$ is not a linear combination of the equations of $\bar{\mathbb{E}}_{\prec(p^i, q^i)}^i$, we proceed with a regularization. Thus $\underline{m}^{i+1} = \underline{m}^i, T^{i+1} = T^i$ and $\mathcal{T}^{i+1} = \mathcal{T}^i$. The combination of the equation $\bar{\Phi}_{\underline{m}^i} H^i \equiv_{(p^i, q^i)} H^i \bar{\Phi}_{\underline{m}^i}$, and the equations of $\bar{\mathbb{E}}_{\prec(p^i, q^i)}^i$ forms an equation system $\bar{\mathbb{E}}^{i+1} : \bar{\Phi}_{\underline{m}^{i+1}} H^{i+1} \equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1} \bar{\Phi}_{\underline{m}^{i+1}}$, where $H^{i+1} = H^i + \emptyset * A_1^i$ by Proposition 2.2.8. And $\bar{\mathbb{E}}^{i+1}$ satisfies assertions (i)–(ii).

2) If $\bar{\mathbb{E}}_{(p^i, q^i)}^i$ is a linear combination of the equations of $\bar{\mathbb{E}}_{\prec(p^i, q^i)}^i$, we proceed with a loop or an edge reduction. There are a pre-minimal algebra $\bar{R}^i = R^i$ in a loop reduction, or $\bar{R}^i = R^i[a_1^i]$ in an edge reduction; a minimal algebra R^{i+1} ; an admissible R^{i+1} - \bar{R}^i -bimodule L^i . Set a size vector $\underline{n} = \vartheta^{i, i+1}(1, \dots, 1)$ over \mathcal{T}^i with $(1, \dots, 1)$ being a size vector over \mathcal{T}^{i+1} , then $\underline{m}^{i+1} = \vartheta^{0i}(\underline{n})$ is

a size vector over \mathcal{T} . Denote by $\bar{\Phi}_{XY} = (u'_{pq})$ for any $(X, Y) \in \mathcal{T}^i \times \mathcal{T}^i$ a submatrix of $\bar{\Phi}_{\underline{m}^i}$, such that $u'_{pq} = u_{pq}$ for $p \in X, q \in Y$, or 0 otherwise. Define a variable matrix over $R^{i+1} \otimes_k R^{i+1}$, and a matrix in $\text{IM}_{\underline{m}^{i+1}}(R^{i+1})$:

$$\begin{aligned} (\bar{\Phi}_{\underline{m}^{i+1}})_{\underline{n}} &= \sum_{(X,Y) \in \mathcal{T}^i \times \mathcal{T}^i} \sum_{1 \leq p \leq n_X, 1 \leq q \leq n_Y} (\mathbf{f}_{Z(X,p)} \otimes_k \mathbf{q}_{Z(Y,q)}) * \bar{\Phi}_{XY}; \\ H_{\underline{n}}^i &= \sum_{X \in \mathcal{T}^i} \sum_{1 \leq p \leq n_X} (\mathbf{f}_{Z(X,p)} \otimes_k \mathbf{q}_{Z(X,p)}) * H_X^i. \end{aligned}$$

where the vertices Z and the matrices $\mathbf{f} \otimes_k \mathbf{e}$ are given before Definition 2.1.2. If the R^i - R^i -quasi-basis $\mathcal{V}^i = \{V_j^i \mid j = 1, \dots, m^i\}$ is a basic solution of $\bar{\mathbb{E}}_{\prec(p^i, q^i)}^i$, then the R^{i+1} - R^{i+1} -quasi-basis $\{(\mathbf{f}_{Z(s(v_j^i), p)} \otimes_k \mathbf{q}_{Z(t(v_j^i), q)}) * V_j^i \mid 1 \leq p \leq n_{s(v_j^i)}, 1 \leq q \leq n_{t(v_j^i)}; j = 1, \dots, m^i\}$ of \mathcal{K}_{11}^{i+1} is a basic system of solutions of the matrix equation $(\bar{\Phi}_{\underline{m}^{i+1}})_{\underline{n}} H_{\underline{n}}^i \equiv_{\prec(p^i, q^i)} H_{\underline{n}}^i (\bar{\Phi}_{\underline{m}^{i+1}})_{\underline{n}}$ partitioned under \mathcal{T}^i , since the (p^i, q^i) -th block is a k -linear combination of the others. In other words, the formal product Π_1^{i+1} of $(\mathcal{K}_{11}^{i+1}, \mathcal{C}_{11}^{i+1})$ is a general solution of the matrix equation.

We may assume that \bar{E}_1 has a R^{i+1} - R^{i+1} -quasi-basis $\{F_1, \dots, F_l\}$ by Definition 2.1.2 (a3), where $0 \leq l \leq \frac{1}{2}sr(r+1)$ after some deletion in Formula (2.2-2) for $a_1^i : X \rightarrow X$, and $F_j L(a_1^i) = L(a_1^i) F_j$; or $l = 0, 1, 2$ after some deletion in Formula (2.2-4) for $a_1^i : X \rightarrow Y$, and $F_1 L(a_1^i) = 0 = L(a_1^i) F_2$. Then either $\mathcal{K}_{10}^{i+1} = \{0\}$, or R^{i+1} - R^{i+1} -quasi-basis of \mathcal{K}_{10}^{i+1} is $\{F'_j = F_j * E_X^i \mid j = 1, \dots, l\}$; or $F'_1 = F_1 * E_X^i$ or $F'_2 = F_2 * E_Y^i$, or both of them given by Construction 2.1.3 (ii), and that of \mathcal{C}_{10}^{i+1} is $\{F_1'^*, \dots, F_l'^*\}$ given by Proof 2) of Proposition 2.1.5. Thus the formal product $\Pi_0^{i+1} = \sum_{j=0}^l F_j'^* * F_j'$ of $(\mathcal{K}_{10}^{i+1}, \mathcal{C}_{10}^{i+1})$ is a general solution of the matrix equation $\Pi_0^{i+1} (L(a_1) * A_1^i) \equiv_{\prec(p^i, q^i)} (L(a_1) * A_1^i) \Pi_0^{i+1}$ partitioned under \mathcal{T}^i , since the (p^i, q^i) -th block is $\sum_1^l (F_j^* * F_j) L(a_1^i) = F_j^* * (L(a_1^i) F_j) = \sum_1^l L(a_1^i) (F_j^* * F_j)$. Define

$$\begin{aligned} \bar{\Phi}_{\underline{m}^{i+1}} &= \Pi_0^{i+1} + (\bar{\Phi}_{\underline{m}^i})_{\underline{n}}, \quad H^{i+1} = H_{\underline{n}}^i + L(a) * A_1^i; \\ \bar{\mathbb{E}}^{i+1} : \bar{\Phi}_{\underline{m}^{i+1}} H^{i+1} &\equiv_{\prec(p^{i+1}, q^{i+1})} H^{i+1} \bar{\Phi}_{\underline{m}^{i+1}}, \end{aligned}$$

where (p^{i+1}, q^{i+1}) is the index followed by the biggest index over \mathcal{T}^{i+1} of the (p^i, q^i) -block partitioned under \mathcal{T}^i . We claim that the formal product $\Pi^{i+1} = \Pi_0^{i+1} + \Pi_1^{i+1}$ of $(\mathcal{K}_1^{i+1}, \mathcal{C}_1^{i+1})$ is a general solution of $\bar{\mathbb{E}}^{i+1}$. First, both left and right sides of each block equation of $(\bar{\Phi}_{\underline{m}^i})_{\underline{n}} (L(a) * A_1^i) \equiv_{\prec(p^i, q^i)} (L(a) * A_1^i) (\bar{\Phi}_{\underline{m}^i})_{\underline{n}}$ partitioned under \mathcal{T}^i are zero blocks, since $(\bar{\Phi}_{\underline{m}^i})_{\underline{n}}$ is a strict upper triangular partitioned matrix and the index of the leading block of $L(a_1^i) * A_1^i$ is (p^i, q^i) . Second, at the left and right sides of each block equation of $\Pi_0^{i+1} H_{\underline{n}}^i \equiv_{\prec(p^i, q^i)} H_{\underline{n}}^i \Pi_0^{i+1}$ under \mathcal{T}^i are equal blocks. In fact, suppose $H_X^i = (h_{\alpha\beta} 1_X)_{\iota^i}$ with $h_{\alpha\beta} = 0$ if $\alpha \notin X$ or $\beta \notin X$, then $(H_{\underline{n}}^i)_X = (H_{X, \alpha\beta})$ with $H_{X, \alpha\beta} = h_{\alpha\beta} \text{diag}(1_{Z(X,1)}, \dots, 1_{Z(X, n_X)})$, therefore $F_j H_{X, \alpha\beta} = H_{X, \alpha\beta} F_j$. And the same assertion is valid for $Y \in \mathcal{T}^i$. Our Theorem follows by induction. \square

For an example, see 2.4.5 (iv) below. $\bar{\mathbb{E}}^i$ in Formula (2.4-4) is also called a *defining system of the pair* $(\mathfrak{A}^i, \mathfrak{B}^i)$. The matrix $\bar{\Phi}_{\underline{m}^i}$ is called a *matrix of dotted elements*. The concept of the dotted elements possesses two folds of meanings: 1) as variables in the equation system $\bar{\mathbb{E}}$; 2) as the elements with a series of linear relations after a sequence of reductions. Different meanings will be used for different cases frequently in Section 4.

Next, we define a deformed system $\bar{\mathbb{F}}^{ri}$ for some fixed $0 < r < s$, which is equivalent to $\bar{\mathbb{E}}^i$. Like the discussion stated before Formula (2.4-3), a matrix equation and a variable matrix of size vector \underline{m}^{ri} over \mathcal{T}^r are defined:

$$\begin{aligned} \bar{\mathbb{F}}^{ri} : \bar{\Psi}_{\underline{m}^{ri}} H_2^i &\equiv_{\prec(p^i, q^i)} \bar{\Psi}_{\underline{m}^{ri}}^0 + H_2^i \bar{\Psi}_{\underline{m}^{ri}}, \\ \bar{\Psi}_{\underline{m}^{ri}}^0 &= H_1^i \bar{\Psi}_{\underline{m}^{ri}} - \bar{\Psi}_{\underline{m}^{ri}} H_1^i, \\ \bar{\Psi}_{\underline{m}^{ri}} &= \sum_{X^r \in \mathcal{T}^r} \bar{w}_{X^r}^{ri} * E_{X^r} + \sum_j \bar{v}_j^{ri} * V_j^r. \end{aligned} \tag{2.4-5}$$

where the definition of $\bar{v}_j^{ri} = (v_{j pq}^{ri})$ and $\bar{w}_{x^r}^{ri} = (w_{x^r pq}^{ri})$ is analogous to that of Formula (2.4-4).

At the end of the subsection, we perform reduction procedure for the matrix bimodule problem given in Example 1.4.5 in order to show some concrete calculations.

Example 2.4.5 (i) Making an edge reduction for the first arrow $a : X \rightarrow Y$ by $a \mapsto G^1 = (1_Z)$, an induced local pair $(\mathfrak{A}^1, \mathfrak{B}^1)$ with $R^1 = k1_Z$; $H^1 = (1_Z) * A$ is obtained.
(ii) Making a loop reduction for $b : Z \rightarrow Z$ by $b \mapsto G^2 = J_2(0)1_X$, an induced local pair $(\mathfrak{A}^2, \mathfrak{B}^2)$ with $R^2 = k1_X$, $H^2 = \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} * A + \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} * B$ is obtained. There are two matrix equalities in the formal equation of the pair $(\mathfrak{A}^2, \mathfrak{B}^2)$:

$$\begin{aligned} & \begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ u_{21}^2 & u_{22}^2 \end{pmatrix} \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11}^2 & v_{12}^2 \\ v_{21}^2 & v_{22}^2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} e & v \\ 0 & e \end{pmatrix}, \\ & \begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^1 & u_{12}^1 \\ u_{21}^1 & u_{22}^1 \end{pmatrix} \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ u_{21}^2 & u_{22}^2 \end{pmatrix} \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} \\ &= \begin{pmatrix} 1_X & 0 \\ 0 & 1_X \end{pmatrix} \begin{pmatrix} v_{11}^2 & v_{12}^2 \\ v_{21}^2 & v_{22}^2 \end{pmatrix} + \begin{pmatrix} 0 & 1_X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_{11}^1 & v_{12}^1 \\ v_{21}^1 & v_{22}^1 \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} e & v \\ 0 & e \end{pmatrix}, \end{aligned}$$

where $(c_{pq})_{2 \times 2}, (d_{pq})_{2 \times 2}$ are splits from c, d respectively, $e \in \mathcal{C}_0^2$ is dual to $E_X = (1_X I_{10}, 1_X I_{10}) \in \mathcal{K}_0^2$, and $v \in \mathcal{C}_1^2$ is dual to $V = \begin{pmatrix} 0 & 1_X \otimes_k 1_X \\ 0 & 0 \end{pmatrix} * E_X \in \mathcal{K}_1^2$ respectively.

(iii) Making a loop mutation $c_{21} \mapsto (x)$, followed by three regularizations, such that $c_{22} \mapsto \emptyset, u_{21}^2 = xv; c_{11} \mapsto \emptyset, v_{21}^2 = vx; c_{12} \mapsto \emptyset, u_{11}^2 = v_{22}^2$, an induced pair $(\mathfrak{A}^3, \mathfrak{B}^3)$ is obtained, and the differentials of the solid arrows in \mathfrak{B}^3 are:

$$\begin{cases} \delta(d_{21}) = xv - vx \\ \delta(d_{22}) = u_{21}^1 + u_{22}^2 - v_{22}^2 - d_{21}v \\ \delta(d_{11}) = u_{11}^2 - v_{11}^2 - v_{21}^1 + vd_{21} \\ \delta(d_{12}) = u_{11}^1 + u_{12}^2 - v_{12}^2 - v_{22}^1 - d_{11}v + vd_{22}. \end{cases}$$

(iv) Finally, we describe the defining systems of Theorem 2.4.1 and 2.4.4 for the pair $(\mathfrak{A}^2, \mathfrak{B}^2)$. Since $(\mathfrak{A}, \mathfrak{B})$ is bipartite, $\mathcal{T} = \mathcal{T}' \times \mathcal{T}''$. Thus $\Phi_{\underline{m}^2} = \Phi_{l^2}^1 \times \Phi_{\underline{n}^2}^2$ and $\bar{\Phi}_{\underline{m}^2} = \bar{\Phi}_{l^2}^1 \times \bar{\Phi}_{\underline{n}^2}^2$, where the size vector $l^2 = (2, 2, 2, 2, 2)$ is over \mathcal{T}' and $\underline{n}^2 = (2, 2, 2, 2, 2)$ over \mathcal{T}'' . Suppose the systems are $\Phi_{l^2}^1 H^2(k) = H^2(k) \Phi_{\underline{n}^2}^2$ given by 2.4.1, and $\bar{\Phi}_{l^2}^1 H^2 = H^2 \bar{\Phi}_{\underline{n}^2}^2$ by 2.4.4 respectively, where

$$\Phi_{l^2}^1 \text{ (or } \bar{\Phi}_{l^2}^1) = \begin{pmatrix} \Phi_0 & 0 & \Phi_1 & \Phi_2 & \Phi_4 \\ & \Phi_0 & \Phi_2 & 0 & \Phi_3 \\ & & \Phi_0 & 0 & \Phi_2 \\ & & & \Phi_0 & \Phi_1 \\ & & & & \Phi_0 \end{pmatrix}, \quad \Phi_{\underline{n}^2}^2 \text{ (or } \bar{\Phi}_{\underline{n}^2}^2) = \begin{pmatrix} \Phi'_0 & 0 & \Phi'_1 & \Phi'_2 & \Phi'_4 \\ & \Phi'_0 & \Phi'_2 & 0 & \Phi'_3 \\ & & \Phi'_0 & 0 & \Phi'_2 \\ & & & \Phi'_0 & \Phi'_1 \\ & & & & \Phi'_0 \end{pmatrix}.$$

Then $\Phi_i = \begin{pmatrix} x_{11}^i & x_{12}^i \\ x_{21}^i & x_{22}^i \end{pmatrix}$ in $\Phi_{l^2}^1$, $\Phi'_i = \begin{pmatrix} y_{11}^i & y_{12}^i \\ y_{21}^i & y_{22}^i \end{pmatrix}$ in $\Phi_{\underline{n}^2}^2$ for $i = 0, 1, 2, 3, 4$ by Theorem 2.4.1; and $\Phi_0 = \begin{pmatrix} 0 & v \\ 0 & 0 \end{pmatrix} = \Phi'_0$ in $\bar{\Phi}_{\underline{m}^2}$, which is obtained from a loop reduction $b \mapsto J_2(0)1_X$; $\Phi_i = \begin{pmatrix} u_{11}^i & u_{12}^i \\ u_{21}^i & u_{22}^i \end{pmatrix}$ in $\bar{\Phi}_{l^2}^1$, $\Phi'_i = \begin{pmatrix} v_{11}^i & v_{12}^i \\ v_{21}^i & v_{22}^i \end{pmatrix}$ in $\bar{\Phi}_{\underline{n}^2}^2$ for $i = 1, 2, 3, 4$ by Theorem 2.4.4.

3 Classification of minimal wild bocses

Based on the well known Drozd's wild configurations, this section is devoted to classifying so-called minimal wild bocses, which are divided into five classes. Then the non-homogeneity

of bocses in the first four classes is proved. But those in the last class have been proved to be strongly homogeneous. Some preliminaries are stated in subsections 3.1 and 3.2.

3.1 An exact structure on representation categories of bocses

In this subsection the concept on exact structure of categories is recalled, especially the exact structure on representation categories of bocses.

Let \mathcal{A} be an additive category with Krull-Schmidt property. We recall from [GR] and [DRSS] the following notions. A pair (ι, π) of composable morphisms

$$(e) \quad M \xrightarrow{\iota} E \xrightarrow{\pi} N \quad (*)$$

in \mathcal{A} is called *exact* if ι is a kernel of π and π is a cokernel of ι .

Let \mathcal{E} be a class of exact pairs which is closed under isomorphisms. The morphisms ι and π appearing in a pair (e) are called an *inflation* and a *deflation* of \mathcal{E} respectively, the pair itself is called a *conflation*, and is denoted by (ι, π) .

Definition 3.1.1 The class \mathcal{E} is said to be an *exact structure* on \mathcal{A} , and $(\mathcal{A}, \mathcal{E})$ an *exact category* if the following axioms are satisfied:

E1 The composition of two deflations is a deflation.

E2 For each φ in $\mathcal{A}(N', N)$ and each deflation π in $\mathcal{A}(E, N)$, there are some E' in \mathcal{A} , an φ' in $\mathcal{A}(E', E)$ and a deflation $\pi' : E' \rightarrow N'$ such that $\pi' \varphi = \varphi' \pi$.

E3 Identities are deflations. If $\varphi \pi$ is a deflation, then so is π .

(Or **E3^{op}** Identities are inflations, if $\iota \varphi$ is an inflation, then so is ι .)

An object P in \mathcal{A} is said to be \mathcal{E} -*projective* (or *projective* for short) if any conflation ending at P is split. Dually an object I in \mathcal{A} is said to be \mathcal{E} -*injective* (or *injective* for short) if any conflation starting at I is split.

Let \mathcal{A} be a Krull-Schmidt category. A morphism $\pi : E \rightarrow N$ in \mathcal{A} is called *right almost split* if it is not a retraction and for any non-retraction $\varphi : L \rightarrow N$, there exists a morphism $\psi : L \rightarrow E$ such that $\varphi = \psi \pi$. It is said that \mathcal{A} has *right almost split morphisms* if for all indecomposable N there exist right almost split morphisms ending at N . Dually, *left almost split morphisms* are defined. It is said that \mathcal{A} has *almost split morphisms* if \mathcal{A} has right and left almost split morphisms.

A morphism $\pi : E \rightarrow N$ is called *right minimal* if every endomorphism $\eta : E \rightarrow E$ with the property that $\pi = \eta \pi$ is an isomorphism. A *left minimal morphism* $\iota : M \rightarrow E$ is defined dually.

Proposition 3.1.2 Suppose that the Krull-Schmidt category \mathcal{A} carries an exact structure \mathcal{E} . Let (e) given in Formula $(*)$ be a conflation. Then the following assertions are equivalent.

- (i) ι is minimal left almost split;
- (ii) π is minimal right almost split;
- (iii) ι is left almost split and π is right almost split.

The conflation (e) in the above proposition is said to be an *almost split conflation*. The exact category $(\mathcal{A}, \mathcal{E})$ is said to *have almost split conflations* if (i) \mathcal{A} has almost split morphisms; (ii) for any indecomposable non-projective N , there exists an almost split conflation (e) ending at N ; (iii) for any indecomposable non-injective M , there exists an almost split conflation (e) starting at M .

Now we turn to the representation category of a bocs. Let $\mathfrak{B} = (\Gamma, \Omega)$ be a bocs with a layer $L = (\Gamma'; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$. From now on it is always assumed that \mathfrak{B} is *triangular on the dotted arrows*, i.e. $\delta(v_j)$ involves only v_1, \dots, v_{j-1} . In particular, the bocs \mathfrak{B} associated to a matrix bimodule problem $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H)$ is triangular by Definition 1.2.1.

The boc $\mathfrak{B}_0 = (\Gamma, \Gamma)$ is called a *principal boc* of \mathfrak{B} . The representation category $R(\mathfrak{B}_0)$ is just the module category of Γ .

Lemma 3.1.3 [O] Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc with a principal boc \mathfrak{B}_0 . Suppose \mathfrak{B} is triangular on the dotted arrows.

(i) If $\iota : M \rightarrow E$ is a morphism of $R(\mathfrak{B})$ with ι_0 injective, then there exist an isomorphism η and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$. Dually, if $\pi : E \rightarrow N$ is a morphism of $R(\mathfrak{B})$ with π_0 surjective, then there exist an isomorphism η and a commutative diagram in $R(\mathfrak{B})$, such that the bottom row is exact in $R(\mathfrak{B}_0)$.

$$\begin{array}{ccccccc} M & \xrightarrow{\iota} & E & & E & \xrightarrow{\pi} & N \\ id \downarrow & & \downarrow \eta & & \eta \downarrow & & \downarrow id \\ 0 & \longrightarrow & M & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & N \longrightarrow 0 \end{array}$$

(ii) If $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota\pi = 0$ is a pair of composable morphisms in $R(\mathfrak{B})$ and $(e_0) : 0 \rightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} N \rightarrow 0$ is exact in the category of vector spaces, then there exists an isomorphism η and a commutative diagram in $R(\mathfrak{B})$:

$$\begin{array}{ccccccc} (e) & & M & \xrightarrow{\iota} & E & \xrightarrow{\pi} & N \\ & & id \downarrow & & \downarrow \eta & & \downarrow id \\ (e') & 0 \longrightarrow & M & \xrightarrow{\iota'} & E' & \xrightarrow{\pi'} & N \longrightarrow 0 \end{array}$$

such that (e') is an exact sequence in $R(\mathfrak{B}_0)$. Moreover, by choosing a suitable basis of M, E', N , we are able to obtain $\iota'_X = (0, I)$ and $\pi'_X = (I, 0)^T$ for all $X \in \mathcal{T}$. \square

Lemma 3.1.4 Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc, which is triangular on the dotted arrows.

(i) $\iota : M \rightarrow E$ is monic in $R(\mathfrak{B})$ if $\iota_0 : M \rightarrow E$ is injective. Dually, $\pi : E \rightarrow N$ is epic in $R(\mathfrak{B})$ if $\pi_0 : E \rightarrow N$ is surjective.

(ii) A pair of composable morphisms $(e) : M \xrightarrow{\iota} E \xrightarrow{\pi} N$ with $\iota\pi = 0$ is exact in $R(\mathfrak{B})$, if $(e_0) : 0 \rightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} N \rightarrow 0$ is exact as a sequence of vector spaces.

Proof (i) If ι_0 is injective, then Lemma 3.1.3 (i) gives a commutative diagram with $\iota' : M \rightarrow E'$ in $R(\mathfrak{B}_0)$. Given any morphism $\varphi : L \rightarrow M$ with $\varphi\iota = 0$, there is $\varphi\iota\eta = \varphi\iota' = 0$. Then $\varphi_0\iota'_0 = 0$ yields $\varphi_0 = 0$. And for any dotted arrow $v_l : X \rightarrow Y$, suppose $\delta(v_l) = \sum_{i,j} u_i \otimes_{\Gamma} u_j$ with $u_i, u_j \in \oplus_{l' < l} \Gamma v_{l'} \Gamma$. There is inductively, $0 = (\varphi\iota')(v_l) = \varphi(v_l)\iota'_Y + \varphi_X\iota'(v_l) + \sum_{i,j} \varphi(u_i)\iota'(u_j) = \varphi(v_l)\iota'_Y$, which yields $\varphi(v_l) = 0$ by the injectivity of ι'_Y . Thus $\varphi = 0$ and ι is monic. The second assertion on π is proved dually.

(ii) It is proved first that ι is the kernel of π . ① If (e_0) is exact, then (i) shows that ι is monic. ② It is known that $\iota\pi = 0$. ③ Lemma 3.1.3 (ii) gives a commutative diagram. If $\varphi : L \rightarrow E$ with $\varphi\pi = 0$, then $\varphi(\eta\pi') = 0$. Let $\xi = \varphi\eta$, then $\xi\pi' = 0$ implies that $\xi_X\pi'_X = 0$ and $\xi(v)\pi'_Y = 0$ for any vertex X and any dotted arrow $v : X \rightarrow Y$. Let $\varphi' : L \rightarrow M$ be given by $\varphi'_X\iota'_X = \xi_X$, $\varphi'(v)\iota'_Y = \xi(v)$, then $\varphi'\iota' = \xi$ is obtained. Thus $\varphi'\iota'\eta^{-1} = \varphi$, i.e. $\varphi'\iota = \varphi$. Therefore ι is a kernel of π . Second, it can be proved dually that π is a cokernel of ι . \square

Let a layered boc $\mathfrak{B} = (\Gamma, \Omega)$ be triangular on the dotted arrows. A class \mathcal{E} of composable morphisms in $R(\mathfrak{B})$ is defined, such that $M \xrightarrow{\iota} E \xrightarrow{\pi} L$ in \mathcal{E} , provided that $\iota\pi = 0$ and

$$0 \longrightarrow M \xrightarrow{\iota_0} E \xrightarrow{\pi_0} L \longrightarrow 0 \quad (3.1-1)$$

is exact as a sequence of vector spaces. It is clear that \mathcal{E} is closed under isomorphisms.

Proposition 3.1.5 [O, Theorem 4.4.1] and [BBP]) Suppose a layered boc \mathfrak{B} is triangular on the dotted arrows. Then the class \mathcal{E} defined by Formula (3.1-1) is an exact structure on $R(\mathfrak{B})$, and $(R(\mathfrak{B}), \mathcal{E})$ is an exact category.

Corollary 3.1.6 ([B1], [O, Lemma 7.1.1]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc.

(i) For any $M \in R(\mathfrak{B})$ with $\dim M = \underline{m}$, if $m_X \neq 0$ for some vertex $X \in \mathcal{T}_1$, then M is neither projective nor injective.

(ii) For any positive integer n , there are only finitely many iso-classes of indecomposable projectives and injectives in $R(\mathfrak{B})$ of dimension at most n .

Remark 3.1.7 ([BCLZ], [O, Definition 4.4.1]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a layered boc, such that $(R(\mathfrak{B}), \mathcal{E})$ is an exact structure. The almost split conflations have been defined in a general exact category, particularly in $R(\mathfrak{B})$.

(i) An indecomposable representation $M \in R(\mathfrak{B})$ is said to be *homogeneous* if there is an almost split conflation $M \xrightarrow{\iota} E \xrightarrow{\pi} M$. The iso-class of M is also said to be *homogeneous*.

(ii) The category $R(\mathfrak{B})$ (or boc \mathfrak{B}) is said to be *homogeneous* if for each positive integer n , almost all (except finitely many) iso-classes of indecomposable representations in $R(\mathfrak{B})$ with size at most n are homogeneous. For example, If \mathfrak{B} is of representation tame type, then $R(\mathfrak{B})$ is homogeneous [CB1].

(iii) The category $R(\mathfrak{B})$ (or boc \mathfrak{B}) is said to be *strongly homogeneous* if there exists neither projectives nor injectives, and all indecomposable representations in $R(\mathfrak{B})$ are homogeneous. For example, if \mathfrak{B} is a local boc with a layer $(R; \omega; a; v)$, $R = k[x, \phi(x)^{-1}]$, and the differential $\delta(a) = xv - vx$. Then $R(\mathfrak{B})$ is strongly homogeneous and representation wild type [BCLZ]. In particular the induced boc given in Example 2.4.5 (iii) is strongly homogeneous.

Note that $(R(\mathfrak{B}), \mathcal{E})$ may not have any almost split conflation. For example, set quiver $Q = a \begin{smallmatrix} \circ \\ \curvearrowright \end{smallmatrix} \cdot \begin{smallmatrix} \circ \\ \curvearrowleft \end{smallmatrix} b$, the path algebra $\Gamma = kQ$, and the principal boc $\mathfrak{B} = (\Gamma, \Gamma)$. Then $R(\mathfrak{B})$ has no almost split conflations, see [V, ZL] for details.

Recalling from [CB1], let $\mathfrak{B} = (\Gamma, \Omega)$ be a minimal boc. Then for any $X \in \mathcal{T}_1$ with $R_X = k[x, \phi_X(x)^{-1}]$, and for any $\lambda \in k$ with $\phi_X(\lambda) \neq 0$, there is an almost split conflation:

$$S(X, 1, \lambda) \xrightarrow{(01)} S(X, 2, \lambda) \xrightarrow{\binom{1}{0}} S(X, 1, \lambda) \quad \text{in } R(\mathfrak{B}), \quad (3.1-2)$$

where $S(X, 1, \lambda)$ (resp. $S(X, 2, \lambda)$) is given by $k \bigcirc_{J_1(\lambda)}$ (resp. $= k^2 \bigcirc_{J_2(\lambda)}$) at X , and $\{0\}$ at other vertices.

3.2 Almost split conflations in the process of reductions

In order to prove the non-homogeneity of some wild bocses, we must understand the behavior of almost split conflations during reduction procedures. We study under what conditions the homogeneous property is preserved after a sequence of reductions in this subsection.

Lemma 3.2.1 [B1] Let $\mathfrak{B}' = (\Gamma', \Omega')$ be the induced boc of a boc \mathfrak{B} given by one of eight reductions in the subsection 2.2, and N' be an indecomposable representation in $R(\mathfrak{B}')$. If N' is non-projective (resp. non-injective) in $R(\mathfrak{B}')$, then so is $\vartheta(N')$ in $R(\mathfrak{B})$.

Lemma 3.2.2 [B1] Let $\mathfrak{B}' = (\Gamma', \Omega')$ be the induced boc of \mathfrak{B} given by one of eight reductions in the subsection 2.2.

(i) If $\iota' : M' \rightarrow E'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\iota') : \vartheta(M') \rightarrow \vartheta(E')$ being a left minimal almost split inflation in $R(\mathfrak{B})$, then so is ι' in $R(\mathfrak{B}')$. Dually if $\pi' : E' \rightarrow N'$ is a morphism in $R(\mathfrak{B}')$ with $\vartheta(\pi') : \vartheta(E') \rightarrow \vartheta(N')$ being a right minimal almost split deflation in $R(\mathfrak{B})$, then so is π' in $R(\mathfrak{B}')$.

(ii) If $(e') : M' \xrightarrow{\ell'} E' \xrightarrow{\pi'} M'$ is a conflation in $R(\mathfrak{B}')$ with $\vartheta(e') : \vartheta(M') \xrightarrow{\vartheta(\ell')} \vartheta(E') \xrightarrow{\vartheta(\pi')} \vartheta(M')$ being an almost split conflation in $R(\mathfrak{B})$, then so is (e') in $R(\mathfrak{B}')$.

Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with trivial R , $M \in R(\mathfrak{A})$ be an indecomposable object of size vector \underline{m} . Set $\mathcal{T}^1 = \{X \in \mathcal{T} \mid m_X \neq 0\}$, suppose \mathfrak{A}^1 is obtained by deleting $\mathcal{T} \setminus \mathcal{T}^1$ from \mathfrak{A} , and $M^1 \in R(\mathfrak{A}^1)$ with $\vartheta^{01}(M^1) \simeq M$. Suppose a sequence of reductions in the sense of Lemma 2.3.2 is given by Theorem 2.3.3 with respect to M^1 :

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^i, \mathfrak{B}^i), (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1}), \dots, (\mathfrak{A}^s, \mathfrak{B}^s). \quad (3.2-1)$$

Then there is some $M^s \in R(\mathfrak{A}^s)$ of sincere size vector \underline{m}^s , such that $\vartheta^{0s}(M^s) \simeq M$.

Theorem 3.2.3 Suppose the first arrow a_1^s is a loop at X^s with $\delta(a_1^s) = 0$; and $M_{X^s}^s = k$, $M^s(a_1^s) = (\lambda)$ in the last term \mathfrak{B}^s of the sequence (3.2-1). If M is homogeneous and $(e) : M \rightarrow E \rightarrow M$ is an almost split conflation in $R(\mathfrak{A})$, then for $i = 1, \dots, s$ there exists an almost split conflation $(e^i) : M^i \rightarrow E^i \rightarrow M^i$ in $R(\mathfrak{A}^i)$, such that $\vartheta^{0i}(e^i) \simeq (e)$.

Proof Induction is used for the proof. The assertion is obviously true for $i = 1$, since the size vector of E is $2\underline{m}$ by Formula (3.1-1). According to Definition 1.3.4 and Formula (2.3-6):

$$M^s = H_{\underline{m}^s}^s(k) + \sum_j M^s(a_j^s) * A_j^s, \quad H_{\underline{m}^s}^s(k) = \sum_{j=1}^{s-1} B^{j+1} * A_1^j.$$

Suppose the assertion is valid for some $1 \leq i < s$. The formula below gives as k -matrices of size vector $\underline{m}^i = \vartheta^{i0}(\underline{m}^s)$:

$$M^i = M^s = H_{\underline{m}^i}^i(k) + B^{i+1} * A_1^i + \sum_{j=2}^{n^i} M^i(a_j^i) * A_j^i. \quad (3.2-2)$$

There exists an object $E^i = H_{2\underline{m}^i}^i(k) + \sum_{j=1}^{m^i} E^i(a_j^i) * A_j^i \in R(\mathfrak{A}^i)$ and an almost split conflation $(e^i) : M^i \xrightarrow{\ell^i} E^i \xrightarrow{\pi^i} M^i$ in $R(\mathfrak{A}^i)$, such that $\vartheta^{0i}(e^i) \simeq (e)$. We now treat the $(i+1)$ -th stage via proving the existence of an isomorphism $\eta : E^i \rightarrow \hat{E}^i$ in $R(\mathfrak{A}^i)$ with $\hat{E}^i(a_1^i) = B^{i+1} \oplus B^{i+1}$.

If this is the case, suppose $a_1^i : X \rightarrow Y$, S_X and S_Y are invertible matrices determined by changing certain rows and columns of $B^{i+1} \oplus B^{i+1}$, such that $S_X^{-1}(B^{i+1} \oplus B^{i+1})S_Y = I_2 \otimes B^{i+1}$, the usual Kronecker product of two matrices. Define a matrix $S = \sum_{Z \in \mathcal{T}^i} S_Z * E_Z$ with $S_Z = I_{m_Z}$ for $Z \in \mathcal{T}^i \setminus \{X, Y\}$. Then there are k -matrices:

$$\begin{aligned} R(\mathfrak{B}^i) \ni \xi(\hat{E}^i) &:= S^{-1} \hat{E}^i S \\ &= H_{2\underline{m}^i}^i(k) + (I_2 \otimes B^{i+1}) * A_1^i + \sum_{j=2}^{n^i} S_{s(a_j^i)}^{-1} \hat{E}^i(a_j^i) S_{t(a_j^i)} * A_j^i \\ &:= H_{2\underline{m}^{i+1}}^{i+1}(k) + \sum_{j=1}^{n^{i+1}} E^{i+1}(a_j^{i+1}) * A_j^{i+1} = E^{i+1} \in R(\mathfrak{A}^{i+1}). \end{aligned}$$

Thus an almost split conflation (e^{i+1}) in $R(\mathfrak{A}^{i+1})$ is obtained by Lemma 3.2.2 (ii), such that $\vartheta^{i,i+1}(e^{i+1}) \simeq (\hat{e}^i) \simeq (e^i)$ via the isomorphisms $E^{i+1} \xrightarrow{\xi^{-1}} \hat{E}^i \xrightarrow{\eta^{-1}} E^i$ in $R(\mathfrak{A}^i)$. Consequently $\vartheta^{0,i+1}(e^{i+1}) \simeq \vartheta^{0i}(e^i) \simeq (e)$.

The existence of such an isomorphism η is established below.

If $\delta(a_1^i) = v_1^i \neq 0$, then $B^{i+1} = M^i(a_1^i) = (0)$. By the proof (i) of Proposition 2.1.8, there exists an isomorphism η , such that $\hat{E}^i = \eta(E^i) \in R(\mathfrak{A}^i)$ with $\hat{E}^i(a_1^i) = (0)$ as desired.

If $\delta(a_1^i) = 0$ in the case of loop or edge reduction, the proof is divided into three parts.

1) We define an object $L^s = H_{2\underline{m}^s}^s(k) + \sum_{i=1}^{n^s} L^s(a_i^s) \in R(\mathfrak{A}^s)$ with $L^s(a_1^s) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. Let $\varphi^s : L^s \rightarrow M^s$ be a morphism in $R(\mathfrak{A}^s)$, such that $\varphi_{Y^s}^s = \begin{pmatrix} I_{Y^s} \\ 0 \end{pmatrix}, \forall Y^s \in \mathcal{T}^s$, and $\varphi^s(v_j^s) = 0$ for any dotted arrow v_j^s in \mathfrak{B}^s . Clearly, φ^s is not a retraction. Thus $\vartheta^{is}(\varphi^s) : \vartheta^{is}(L^s) \mapsto \vartheta^{is}(M^s) = M^i$ is not a retraction, since the functor ϑ^{is} is fully faithful.

2) Because $\vartheta^{is}(L^s) = H_{2m^i}^i(k) + (I_2 \otimes B^{i+1}) * A_1^i + \sum_{j=2}^{n^i} \vartheta^{is}(L^s)(a_j^i) * A_j^i$, it is possible to construct an object L^i with $L^i(a_1^i) = B^{i+1} \oplus B^{i+1}$ by changing certain rows and columns in $I_2 \otimes B^{i+1}$, and an isomorphism $\vartheta^{is}(L^s) \xrightarrow{\zeta} L^i$. Let $\varphi^i = \vartheta^{is}(\varphi^s)\zeta^{-1} : L^i \rightarrow M^i$, which is not a retraction by 1). Thus there is a lifting $\tilde{\varphi}^i : L^i \rightarrow E^i$ of φ^i with $\varphi^i = \tilde{\varphi}^i \pi^i$, since $\pi^i : E^i \rightarrow M^i$ is right almost split in $R(\mathfrak{A}^i)$ by the assumption on (e^i) . The triangle and the square below are both commutative:

$$\begin{array}{ccc} & L^i & \\ \tilde{\varphi}^i \swarrow & & \searrow \varphi^i \\ E^i & \xrightarrow{\pi^i} & M^i \end{array} \quad \begin{array}{ccc} L_X^i & \xrightarrow{L^i(a_1^i)} & L_Y^i \\ \tilde{\varphi}_X^i \downarrow & & \downarrow \tilde{\varphi}_Y^i \\ E_X^i & \xrightarrow{E^i(a_1^i)} & E_Y^i \end{array}$$

3) According to Lemma 3.1.3, it may be assumed that the sequence $(e^i) \in R(\mathfrak{B}_0^i)$ with $\iota_Z^i = (0 \ I_Z), \pi_Z^i = \begin{pmatrix} I_Z \\ 0 \end{pmatrix}, \forall Z \in \mathcal{T}^i$, then $E^i(a_j^i) = \begin{pmatrix} M^i(a_j^i) & K_j^i \\ 0 & M^i(a_j^i) \end{pmatrix}$. The commutative triangle forces $\tilde{\varphi}_Z = \begin{pmatrix} I_Z & C_Z \\ 0_Z & D_Z \end{pmatrix}$ for each $Z \in \mathcal{T}^i$. The commutative square for $j = 1$ gives an equality

$$\begin{pmatrix} I_X & C_X \\ & D_X \end{pmatrix} \begin{pmatrix} B^{i+1} & K_1^i \\ & B^{i+1} \end{pmatrix} = \begin{pmatrix} B^{i+1} & \\ & B^{i+1} \end{pmatrix} \begin{pmatrix} I_Y & C_Y \\ & D_Y \end{pmatrix}.$$

Let $\hat{E}^i = \{\hat{E}_Z^i \mid \dim(E_Z^i) = 2m_Z^i, Z \in \mathcal{T}^i\}$ be a set of vector spaces. Define a set of maps $\eta : E^i \rightarrow \hat{E}^i$, such that $\eta_X = \begin{pmatrix} I_X & C_X \\ 0 & I_X \end{pmatrix}$, $\eta_Y = \begin{pmatrix} I_Y & C_Y \\ 0 & I_Y \end{pmatrix}$, and $\eta_Z = I_{2m_Z^i}$ for $Z \in \mathcal{T}^i \setminus \{X, Y\}$; $\eta(v_j) = 0$ for any $j = 1, \dots, m^i$. Let $\hat{E}^i(a_j^i) = \eta_{s(a_j^i)} E^i(a_j^i) \eta_{t(a_j^i)}^{-1}$ for $j = 1, \dots, n^i$, an object $\hat{E}^i = \eta E^i \eta^{-1} \simeq E^i$ in $R(\mathfrak{A}^i)$ with $\hat{E}^i(a_1^i) = B^{i+1} \oplus B^{i+1}$ is obtained as desired. \square

Suppose in the sequence below, the first part from the 0-th pair up to the s -pair is given by Formula (3.2-1) with respect to the indecomposable object $M = \vartheta^{0i}(M^s) \in R(\mathfrak{A})$:

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^s, \mathfrak{B}^s), (\mathfrak{A}^{s+1}, \mathfrak{B}^{s+1}) \dots, (\mathfrak{A}^\varepsilon, \mathfrak{B}^\varepsilon), \dots, (\mathfrak{A}^\tau, \mathfrak{B}^\tau). \quad (3.2-3)$$

Firstly, it is assumed that in the sequence (3.2-3), \mathfrak{B}^s is local, $\mathcal{T}^s = \{X\}$; \mathfrak{B}^{s+1} is induced from \mathfrak{B}^s by a loop mutation; the reduction from \mathfrak{B}^i to \mathfrak{B}^{i+1} is given by a localization followed by a regularization, such that $R^{i+1} = k[x, \phi^{i+1}(x)^{-1}]$ for $s < i < \tau$; and \mathfrak{B}^τ is minimal.

Corollary 3.2.4 Suppose M^τ is an object of $R(\mathfrak{B}^\tau)$ with $M_X^\tau = k, M^\tau(x) = (\lambda), \phi^\tau(\lambda) \neq 0$. If $\vartheta^{0\tau}(M^\tau) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation (e) , then there exists an almost split conflation (e_λ^τ) given by Formula (3.1-2) in $R(\mathfrak{B}^\tau)$, such that $\vartheta^{0\tau}(e_\lambda^\tau) \simeq (e)$.

Proof Set $M^s = \vartheta^{s\tau}(M^\tau)$, then $M^s(a_1^s) = (\lambda)$. Theorem 3.2.3 gives an almost split conflation (e^s) in $R(\mathfrak{B}^s)$ with $\vartheta^{0s}(e^s) \simeq (e)$. Furthermore, $R(\mathfrak{A}^{s+1})$ is equivalent to $R(\mathfrak{A}^s)$, and for $i > s$, $R(\mathfrak{A}^{i+1})$ is equivalent to a subcategory of $R(\mathfrak{A}^i)$ consisting of the objects M^i with $M^i(x) = (\lambda), \phi^{i+1}(x)$. The assertion follows by the fact that $\phi^{i+1}(x) \mid \phi^\tau(x)$, and induction on $i = s+1, \dots, \tau-1$. \square

Secondly, it is assumed that in the sequence (3.2-3) the boc \mathfrak{B}^s has two vertices X, Y , and the first arrow $a_1^s : X \rightarrow X$ with $\delta(a_1^s) = 0$. The boc \mathfrak{B}^{s+1} is induced from \mathfrak{B}^s by a loop mutation $a_1^s \mapsto (x)$, such that a_j^{s+1} is either a loop at X , or an edge from X to Y for $j = 1, \dots, \tau - s$. In particular, there exists a certain index $s < \varepsilon < \tau$, such that $a_{\varepsilon-s}^{s+1} : X \rightarrow Y$ is an edge. The reduction from \mathfrak{B}^i to \mathfrak{B}^{i+1} is given by one of the following three cases:

- (i) when $s < i < \varepsilon$, if $a_1^i : X \rightarrow X$, a localization followed by a regularization are made with $R^{i+1} = k[x, \phi^{i+1}(x)^{-1}]$; if $a_1^i : X \rightarrow Y$, a regularization, or a reduction given by proposition 2.2.6 is made;

- (ii) when $i = \varepsilon$, a reduction given by proposition 2.2.7 is made, the induced boc $\mathfrak{B}^{\varepsilon+1}$ is local with a vertex Z ;
- (iii) when $\varepsilon < i < \tau$, then $a_1^i : Z \rightarrow Z$, a localization followed by a regularization are made. Finally, \mathfrak{B}^τ is minimal with $R^\tau = k[x, \phi^\tau(x)^{-1}]$.

Corollary 3.2.5 Suppose M^τ is an object of $R(\mathfrak{B}^\tau)$, $M_Z^\tau = k$, $M^\tau(z) = (\lambda)$ with $\phi^\tau(\lambda) \neq 0$. If $\vartheta^{0\tau}(M^\tau) = M \in R(\mathfrak{B})$ is homogeneous with an almost split conflation (e) , then there exists an almost split conflation (e_λ^τ) given by Formula (3.1-2) in $R(\mathfrak{B}^\tau)$, such that $\vartheta^{0\tau}(e_\lambda^\tau) \simeq (e)$.

Proof Set $M^s = \vartheta^{st}(M^t)$, then $M_X^s = k$, $M_Y^s = k$, and $M^s(a_1^s) = (\lambda)$. Theorem 3.2.3 gives an almost split conflation $(e^s) : M^s \rightarrow E^s \rightarrow M^s$ in \mathfrak{B}^s with $\vartheta^{0s}(e^s) \simeq (e)$. Since $R(\mathfrak{B}^{s+1})$ is equivalent to $R(\mathfrak{B}^s)$, we may suppose for some $i > s$, there exists an almost split conflation

$$(e^i) : M^i \xrightarrow{\iota^i} E^i \xrightarrow{\pi^i} M^i \in R(\mathfrak{B}^i) \quad \text{with} \quad M^i = \vartheta^{it}(M^t), \quad \vartheta^{si}(e^i) \simeq (e^s).$$

An almost split conflation (e^{i+1}) in $R(\mathfrak{B}^{i+1})$ with $\vartheta^{i,i+1}(e^{i+1}) \simeq (e^i)$ will be constructed according to cases (i)–(iii) stated before the corollary.

(i) A regularization for an edge gives an equivalence $R(\mathfrak{B}^{i+1}) \simeq R(\mathfrak{B}^i)$. And the proof of a regularization for a loop is similar to that of Corollary 3.2.4. Suppose $\delta(a_1^i) = 0$ in \mathfrak{B}^i , and a reduction of Proposition 2.2.6 is made by $a_1^i \mapsto (0)$. Then $M^i(a_1^i)$ must be (0) . By Lemma 3.1.3 (ii), it may be assumed that $\iota^i = (0 \ 1)$, $\pi^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, thus $E^i(a_1^i) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Define an object $L \in R(\mathfrak{B}^i)$ of size $2\underline{m}^i$, such that $L(x) = J_2(\lambda)$, $L(a_j^i) = 0, \forall j$; and a morphism $g : L \rightarrow M^i$, such that $g_X = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = g_Y$, $g(v) = 0$ for any dotted arrow v . Then g is not a split epimorphism. Thus there exists a lifting $\tilde{g} : L \rightarrow E^i$ with $\tilde{g}_X = \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix}$, $\tilde{g}_Y = \begin{pmatrix} 1 & c' \\ 0 & d' \end{pmatrix}$. Since \tilde{g} is a morphism, $L^i(a_1^i)\tilde{g}_X = \tilde{g}_Y E^i(a_1^i)$, which leads to $0 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Therefore $b = 0$, $E^i(a_1^i) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Set $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $E^{i+1}(x) = E^i(x)$, $E^{i+1}(a_{j-1}^{i+1}) = E^i(a_j^i)$ for $j = 2, \dots, n^i$, then $\vartheta^{i,i+1}(E^{i+1}) = E^i$.

(ii) Proposition 2.2.7 ensures a possibility that $M^\varepsilon(a_1^\varepsilon) = (1)$, so $E^\varepsilon(a_1^\varepsilon) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Define a set of matrices: $\{\xi_Y = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \xi_X = I_2, \xi(v_j^\varepsilon) = 0\}$, and an object $\bar{E}^\varepsilon \in R(\mathfrak{B}^\varepsilon)$: $\bar{E}_X^\varepsilon = k^2 = \bar{E}_Y^\varepsilon$, $\bar{E}^\varepsilon(x) = E^\varepsilon(x)$, $\bar{E}^\varepsilon(a_j^\varepsilon) = \xi_{s(a_j^\varepsilon)}^{-1} E^\varepsilon(a_j^\varepsilon) \xi_{t(a_j^\varepsilon)}$, then $\bar{E}^\varepsilon(a_1^\varepsilon) = I_2$. Let $E^{\varepsilon+1} \in R(\mathfrak{B}^{\varepsilon+1})$ with $E_Z^{\varepsilon+1} = k^2$; $E^{\varepsilon+1}(z) = E^\varepsilon(x)$, $E^{\varepsilon+1}(a_{j-1}^{\varepsilon+1}) = \bar{E}^\varepsilon(a_j^\varepsilon)$ for $j > 1$. Then $\vartheta^{\varepsilon,\varepsilon+1}(E^{\varepsilon+1}) = \bar{E}^\varepsilon \simeq E^\varepsilon$ in $R(\mathfrak{B}^\varepsilon)$.

(iii) Since $\phi^\tau(\lambda) \neq 0$ and $\phi^{i+1}(x) \mid \phi^t(x)$, $\phi^{i+1}(\lambda) \neq 0$. There is an object $E^{i+1} \in R(\mathfrak{B}^{i+1})$ with $\vartheta^{i,i+1}(E^{i+1}) \simeq E^i$.

By induction, there is $(e_\lambda^\tau) \in R(\mathfrak{B}^\tau)$ with $\vartheta^{s\tau}(e_\lambda^\tau) \simeq e_\lambda^s$, thus $\vartheta^{0\tau}(e_\lambda^\tau) \simeq (e)$. \square

Lemma 3.2.6 (i) Suppose that $f(x, y) = \sum_{i,j \geq 0} \alpha_{ij} x^i y^j \in k[x, y]$ with $f(\lambda, \mu) \neq 0$. Let W_λ, W_μ be Weyr matrices of size m, n and eigenvalues λ, μ respectively, and $V = (v_{ij})_{m \times n}$ with $\{v_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ being k -linearly independent. Let $f(W_\lambda, W_\mu)V = \sum_{i,j \geq 0} \alpha_{ij} W_\lambda^i V W_\mu^j = (u_{ij})_{m \times n}$. Then $\{u_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is also k -linearly independent.

(ii) Let \mathfrak{B} be a boc with $R = R_X \times R_Y$, where $R_X = k[x, \phi_X(x)^{-1}]$, $R_Y = k[y, \phi_Y(y)^{-1}]$, and $a_i : X \rightarrow Y$. Define $\delta^0(a_i)$ to be a *part of* $\delta(a_i)$ without terms involving any solid arrow. It is possible that $X = Y$, in this case x stands for the multiplying a dotted arrow from the left and y from the right. Suppose

$$\begin{cases} \delta^0(a_1) = f_{11}(x, y)v_1 \\ \delta^0(a_2) = f_{21}(x, y)v_1 + f_{22}(x, y)v_2, \\ \dots \dots \\ \delta^0(a_n) = f_{n1}(x, y)v_1 + f_{n2}(x, y)v_2 + \dots f_{nn}(x, y)v_n, \end{cases}$$

where $f_{ii}(x, y) \in R_X \times R_Y$ are invertible for $i = 1, 2, \dots, n$. If $x \mapsto W_X$ of size m with eigenvalue λ and $\phi_X(\lambda) \neq 0$, $y \mapsto W_Y$ of size n with eigenvalue μ and $\phi_Y(\mu) \neq 0$, then the solid arrows splitting from a_1, \dots, a_n are all going to \emptyset by regularizations in further reductions.

Proof (i) Since $u_{ij} = f(\lambda, \mu)v_{ij} + \sum_{(i', j') \prec (i, j)} f_{i'j'}(\lambda, \mu)v_{i'j'}$ with $f_{i'j'}(x, y) \in k[x, y]$, the assertion follows by induction on the ordered index set $\{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.
(ii) The conclusion follows by (i) inductively on $1, \dots, n$. \square

3.3 Minimal wild bocses

In this subsection five classes of minimal wild bocses is defined in order to prove the main theorem. Our classification relies on the Drozd's wild configurations with refinements at some last reduction stages.

Proposition 3.3.1 ([D1],[CB1]) Let $\mathfrak{B} = (\Gamma, \Omega)$ be a bocs with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ and suppose $a_1 : X \rightarrow Y$. If \mathfrak{B} is of representation wild type, then it is bound to meet one of the following configurations at some stage of reductions:

Case 1 $X \in \mathcal{T}_1, Y \in \mathcal{T}_0$ (or dually $X \in \mathcal{T}_0, Y \in \mathcal{T}_1$), $\delta(a_1) = 0$.

Case 2 $X, Y \in \mathcal{T}_1$ (possibly $X = Y$), $\delta(a_1) = f(x, y)v_1$ for some non-invertible $f(x, y)$ in $k[x, y, \phi_X(x)^{-1}, \phi_Y(y)^{-1}]$. \square

Some notations will be fixed first before the classification. There is a decomposition for any non-zero polynomial $f(x, y) \in k[x, y]$:

$$f(x, y) = \alpha(x)h(x, y)\beta(y), \quad \text{where } \alpha(x) \in k[x], \beta(y) \in k[y]; \quad (3.3-1)$$

and every irreducible factor of $h(x, y)$ contains both x and y , or $h(x, y) \in k^*$ with $k^* = k \setminus \{0\}$. Sometimes \bar{x} is used instead of y .

Let $k(x, y, z)$ be the fractional field of the polynomial ring $k[x, y, z]$ of three indeterminates. Consider a vector space \mathcal{S} generated by the dotted arrows $\{v_1, \dots, v_m\}$ of a bocs \mathfrak{B} over $k(x, y, z)$. Suppose there is a linear combination:

$$G = f_1(x, y, z)v_1 + \dots + f_m(x, y, z)v_m, \quad f_i(x, y, z) \in k[x, y, z]. \quad (*)$$

Let $h(x, y, z)$ be the greatest common factor of f_1, \dots, f_m , then $f_1/h, \dots, f_m/h$ are co-prime, and $G = h \sum_{i=1}^m (f_i/h)v_i$. There exists some $s_i \in k[x, y, z]$ for $i = 1, \dots, m$, such that $\sum_{i=1}^m s_i(f_i/h) = c(x, y) \in k[x, y]$. Since $S = k[x, y, z, c(x, y)^{-1}]$ is a Hermite ring, there exists some invertible $F(x, y, z) \in \text{IM}_m(S)$ with the first column $(f_1/h, \dots, f_m/h)$. A base change of the form $(w_1, \dots, w_m) = (v_1, \dots, v_m)F$ is made, thus $G = h(x, y, z)w_1$.

Classification 3.3.2 Let \mathfrak{B}^0 be a wild bocs given by Proposition 3.3.1. Then we are bound to meet an induced bocs \mathfrak{B} with a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ in one of the five classes at some stage of reductions. And a bocs in those classes is said to be *minimal wild*, which might be written briefly by MW.

Suppose the bocs \mathfrak{B} has two vertices $\mathcal{T} = \{X, Y\}$, such that the induced local bocs \mathfrak{B}_X is tame infinite with $R_X = k[x, \phi_X(x)^{-1}]$.

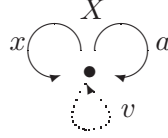
MW1 \mathfrak{B}_Y is finite with $R_Y = k1_Y$, and $\delta(a_1) = 0$:

$$x \bigcirc X \xrightarrow{a_1} Y.$$

MW2 \mathfrak{B}_Y is tame infinite with $R_Y = k[y, \phi_Y(y)^{-1}]$, and $\delta(a_1) = f(x, y)v_1$, such that $f(x, y)$ is non-invertible in $k[x, y, \phi_X(x)^{-1}, \phi_Y(y)^{-1}]$:

$$x \bigcirc X \xrightarrow{a_1} Y \bigcirc y.$$

Suppose now we have a local bocs \mathfrak{B} with $R = k[x, \phi(x)^{-1}]$:



MW3 The differential δ^0 of the solid arrows of \mathfrak{B} is given by

$$\begin{cases} \delta^0(a_1) &= f_{11}(x, \bar{x})w_1, \\ \dots & \dots \\ \delta^0(a_n) &= f_{n1}(x, \bar{x})w_1 + \dots + f_{nn}(x, \bar{x})w_n, \end{cases} \quad (3.3-2)$$

where w_i is obtained by base changes; $f_{ii}(x, \bar{x}) = \alpha_{ii}(x)h_{ii}(x, \bar{x})\beta_{ii}(\bar{x})$ by Formula (3.3-1), such that $h_{ii}(x, x)$ is invertible in $k[x, \phi(x)^{-1}]$ for $1 \leq i \leq n$; and there is some minimal $1 \leq s \leq n$, such that $f_{ss}(x, \bar{x})$ is non-invertible in $k[x, \bar{x}, \phi(x)^{-1}, \phi(\bar{x})^{-1}]$.

Suppose there exists some $1 \leq n_1 \leq n$, such that:

$$\begin{cases} \delta^0(a_1) &= f_{11}(x, \bar{x})w_1, \\ \dots & \dots \\ \delta^0(a_{n_1-1}) &= f_{n_1-1,1}(x, \bar{x})w_1 + \dots + f_{n_1-1,1}(x, \bar{x})w_{n_1-1}, \\ \delta^0(a_{n_1}) &= f_{n_1,1}(x, \bar{x})w_1 + \dots + f_{n_1,n_1-1}(x, \bar{x})w_{n_1-1} + f_{n_1,n_1}(x, \bar{x})\bar{w}, \end{cases} \quad (3.3-3)$$

where $f_{ii}(x, x)$ for $1 \leq i < n_1$ are invertible in $k[x, \phi(x)^{-1}]$; $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1,n_1}(x, x) = 0$. Denote by x_1 the solid arrow a_{n_1} , there exists a polynomial $\psi(x, x_1)$ being divided by $\phi(x)$. Write δ^1 the part of differential δ by deleting all the monomials involving any solid arrow except x, x_1 . Suppose the further unraveling for x is restricted to $x \mapsto (\lambda)$ with $\psi(\lambda, x_1) \neq 0$. Then

$$\begin{cases} \delta^1(a_{n_1+1}) &= K_{n_1+1} + f_{n_1+1,n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1}, \\ \dots & \dots \\ \delta^1(a_n) &= K_n + f_{n,n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1} + \dots + f_{nn}(x, x_1, \bar{x}_1)w_n, \end{cases} \quad (3.3-4)$$

where $K_i = \sum_{j=1}^{n_1-1} f_{ij}(x, x_1, \bar{x}_1)w_j$, w_i are given by the base changes described below Formula (*) inductively, and $f_{ii}(x, x_1, \bar{x}_1)$ are invertible in $k[x, x_1, \bar{x}_1, \psi(x, x_1)^{-1}, \psi(x, \bar{x}_1)^{-1}]$ for $n_1 < i \leq n$.

MW4 $\bar{w} = 0$, or $\bar{w} \neq 0$ but $(x - \bar{x})^2 \mid f_{n_1 n_1}(x, \bar{x})$ in Formula (3.3-3).

MW5 $\bar{w} \neq 0$ and $(x - \bar{x})^2 \nmid f_{n_1 n_1}(x, \bar{x})$ in Formula (3.3-3). \square

The proof of Classification 3.3.2 depends on Classification 3.3.5 of local bocses at the end of the subsection, while the proof of 3.3.5 is based on formulae (3.3-2)-(3.3-9) and Lemma 3.3.3-3.3.4 below.

Let \mathfrak{B} be a local boc having a layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$. If $R = k1_X$ is trivial, then the differentials of the solid arrows have two possibilities. First,

$$\begin{cases} \delta^0(a_1) &= f_{11}w_1, \\ \dots & \dots \\ \delta^0(a_n) &= f_{n1}w_1 + \dots + f_{nn}w_n, \end{cases} \quad (3.3-5)$$

where $f_{ij} \in k, f_{ii} \neq 0$ for $1 \leq i \leq n$. Second, there exists some $1 \leq n_0 \leq n$, such that:

$$\begin{cases} \delta^0(a_1) &= f_{11}w_1, \\ \dots & \dots \\ \delta^0(a_{n_0-1}) &= f_{n_0-1,1}w_1 + \dots + f_{n_0-1,n_0-1}w_{n_0-1}, \\ \delta^0(a_{n_0}) &= f_{n_0,1}w_1 + \dots + f_{n_0,n_0-1}w_{n_0-1}, \end{cases} \quad (3.3-6)$$

where $f_{ij} \in k, f_{ii} \neq 0$ for $1 \leq i < n_0$. Set $a_i \mapsto \emptyset, i = 1, \dots, n_0 - 1$, by a series of regularization, then $a_{n_0} \mapsto (x)$ by a loop mutation, an induced local boc \mathfrak{B}' is obtained.

Without loss of generality, the boc \mathfrak{B}' may still be denoted by \mathfrak{B} with a layer L , but $R = k[x]$. The differentials δ^0 have again two possibilities. The first one is given by Formula (3.3-2), such that $h_{ii}(x, x) \neq 0$, i.e. $(x - \bar{x}) \nmid f_{ii}(x, \bar{x})$ for $i = 1, \dots, n$. Define a polynomial:

$$\phi(x) = \prod_{i=1}^n c_i(x) h_{ii}(x, x), \quad (3.3-7)$$

where $c_i(x)$ appears at the localization in order to do a base change before the i -th step of a regularization.

Lemma 3.3.3 Let \mathfrak{B} be a boc \mathfrak{B} given by Formula (3.3-2) with a polynomial $\phi(x)$ given by Formula (3.3-7). There exist two cases:

- (i) $f_{ii}(x, \bar{x})$ are invertible in $k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ for $1 \leq i \leq n$;
- (ii) $f_{ss}(x, \bar{x})$ is not invertible in $k[x, \bar{x}, \phi(x)^{-1}\phi(\bar{x})^{-1}]$ for some minimal $1 \leq s \leq n$. \square

The second possibility of the differential δ^0 in the case of $R = k[x]$ is given by Formula (3.3-3) for some fixed $1 \leq n_1 \leq n$, such that $f_{ii}(x, x) \neq 0$ for $1 \leq i < n_1$, and $\bar{w} = 0$, or $\bar{w} \neq 0$ but $f_{n_1, n_1}(x, x) = 0$. Let

$$\phi(x) = \begin{cases} \prod_{i=1}^{n_1-1} c_i(x) f_{ii}(x, x), & \bar{w} = 0; \\ c_{n_1}(x) \prod_{i=1}^{n_1-1} c_i(x) f_{ii}(x, x), & \bar{w} \neq 0, \end{cases} \quad (3.3-8)$$

Thus, under the restriction $x \mapsto (\lambda), \phi(\lambda) \neq 0$, an induced boc given by regularizations with the first arrow a_{n_1} is obtained. There are two possibilities in the further reductions. The first possibility is given by Formula (3.3-4), such that $f_{ii}(x, x_1, x_1) \neq 0$ for $n_1 < i \leq n$. There is a sequence of localizations given by the polynomials $c_i(x, x_1)$ appeared before the base changes in order to do regularizations. Let

$$\psi(x, x_1) = \phi(x) \prod_{i=n_1+1}^n c_i(x, x_1) f_{ii}(x, x_1, x_1), \quad (3.3-9)$$

Lemma 3.3.4 Let the differentials in the boc \mathfrak{B} be given by Formulae (3.3-3)–(3.3-4) with polynomials $\phi(x)$ in (3.3-8), and $\psi(x, x_1)$ in (3.3-9). There exist two cases.

(i) There exists some $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, and a minimal $n_1 + 1 \leq s \leq n$, such that $f_{ss}(\lambda, x_1, \bar{x}_1)$ is non-invertible in $k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$, i.e., after making an unraveling $x \mapsto (\lambda)$, followed by a series of regularizations $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \dots, n_1 - 1$, the induced local boc $\mathfrak{B}_{(\lambda)}$ with $R_{(\lambda)} = k[x_1, \psi(\lambda, x_1)^{-1}]$ is in case (ii) of Lemma 3.3.3.

(ii) For any $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, $f_{ii}(\lambda, x_1, \bar{x}_1)$ are invertible for $n_1 < i \leq n$ in $k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$, i.e., the induced boc $\mathfrak{B}_{(\lambda)}$ with $R_{(\lambda)} = k[x_1, \psi(\lambda, x_1)^{-1}]$ is in case (i) of 3.3.3.

Case (ii) is equivalent to (ii)': $f_{ii}(x, x_1, \bar{x}_1)$ are invertible in $k[x, x_1, \bar{x}_1, \psi(x, x_1)^{-1}, \psi(x, \bar{x}_1)^{-1}]$ for $n_1 < i \leq n$.

Proof It is only need to prove the equivalence of (ii) and (ii)'.

(ii) \implies (ii)' If there exists some $n_1 < s \leq n$ with $f_{ss}(x, x_1, \bar{x}_1)$ non-invertible, then it contains a non-trivial factor $g(x, x_1, \bar{x}_1)$ coprime to $\psi(x, x_1)\psi(x, \bar{x}_1)$. Consider the variety $V = \{(\alpha, \beta, \gamma) \in k^3 \mid g(\alpha, \beta, \gamma) = 0, \psi(\alpha, \beta)\psi(\alpha, \gamma) = 0\}$. Since $\dim(V) \leq 1$, there exists a co-finite subset $\mathcal{L} \subset k$, such that $\forall \lambda \in \mathcal{L}$, the plane $x = \lambda$ of k^3 intersects V at only a finite number of points. Thus $g(\lambda, x, \bar{x}_1)$ and $\psi(\lambda, x_1)\psi(\lambda, \bar{x}_1)$ are coprime. Consequently $g(\lambda, x, \bar{x}_1)$, thus $f_{ss}(\lambda, x_1, \bar{x}_1)$ is not invertible in $k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$.

(ii)' \implies (ii) If $f_{ii}(x, x_1, \bar{x}_1)$ is invertible in $k[x, x_1, \bar{x}_1, \psi(x, x_1)^{-1}\psi(x, \bar{x}_1)^{-1}]$, then for any $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, $f_{ii}(\lambda, x_1, \bar{x}_1)$ is invertible in $k[x_1, \bar{x}_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$. \square

The second possibility of δ^1 is: there exists some n_2 with $n_1 < n_2 \leq n$, such that

$$\begin{cases} \delta^1(a_{n_1+1}) = K_{n_1+1} + f_{n_1+1, n_1+1}(x, x_1, \bar{x}_1)w_{n_1+1}, \\ \dots \dots \\ \delta^1(a_{n_2-1}) = K_{n_2-1} + \dots + f_{n_2-1, n_2-1}(x, x_1, \bar{x}_1)w_{n_2-1}, \\ \delta^1(a_{n_2}) = K_{n_2} + \dots + f_{n_2, n_2-1}(x, x_1, \bar{x}_1)w_{n_2-1} + f_{n_2, n_2}(x, x_1, \bar{x}_1)\bar{w}', \end{cases} \quad (3.3-10)$$

where $K_i = \sum_{j=1}^{n_1-1} f_{ij}(x, x_1, \bar{x}_1) w_j$ for $n_1 < i \leq n_2$, $f_{ii}(x, x_1, x_1) \neq 0$ for $n_1 < i < n_2$, and $\bar{w}' = 0$, or $\bar{w}' \neq 0$ but $f_{n_2, n_2}(x, x_1, x_1) = 0$. Define a polynomial

$$\psi_1(x, x_1) = \begin{cases} \phi(x) \prod_{i=n_1+1}^{n_2-1} c_i(x, x_1) f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' = 0; \\ c_{n_2}(x, x_1) \phi(x) \prod_{i=n_1+1}^{n_2-1} c_i(x, x_1) f_{ii}(x, x_1, x_1), & \text{if } \bar{w}' \neq 0. \end{cases} \quad (3.3-11)$$

Suppose a boc \mathfrak{B} is med, the differential of which is given by Formula (3.3-3) and (3.3-10) with a polynomial $\psi_1(x, x_1)$ of (3.3-11). Fix any $\lambda_0 \in k$ with $\psi_1(\lambda_0, x_1) \neq 0$, there is an induced boc $\mathfrak{B}_{(\lambda_0)}$ with $R_{(\lambda_0)} = k[x_1, \psi_1(\lambda_0, x_1)^{-1}]$ given by an unraveling $x \mapsto (\lambda_0)$, and then a series of regularization $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \dots, n_1 - 1$. There exist three cases:

1) $\mathfrak{B}_{(\lambda_0)}$ is in case (i) of Lemma 3.3.4, then there exists some λ_1 with $\psi(\lambda_0, \lambda_1) \neq 0$, such that after sending $x_1 \mapsto (\lambda_1)$ by an unraveling, followed by a series of regularizations, the induced boc $\mathfrak{B}_{(\lambda_0, \lambda_1)}$ satisfies Lemma 3.3.3 (ii);

2) $\mathfrak{B}_{(\lambda_0)}$ is in case (ii)' of Lemma 3.3.4;

3) $\mathfrak{B}_{(\lambda_0)}$ is in the case of Formulae (3.3-3) and (3.3-10).

In case 3), the above procedure is repeated once again for $\mathfrak{B}_{(\lambda_0)}$. By induction on the number of the finitely many solid arrows, the case 1) or case 2) is finally reached.

Classification 3.3.5 Let \mathfrak{B} be a local boc with R trivial, there exist four cases:

(i) \mathfrak{B} has Formula (3.3-5).

(ii) \mathfrak{B} has Formula (3.3-6). And by a series of regularizations $a_i \mapsto \emptyset, i = 1, \dots, n_0 - 1$, followed by a loop mutation $a_{n_0} \mapsto (x)$, the induced boc \mathfrak{B}' with a polynomial (3.3-7) is in case (i) of Lemma 3.3.3.

(iii) \mathfrak{B} has an induced local boc $\mathfrak{B}_{(\lambda_0, \lambda_1, \dots, \lambda_l)}$ for some $l < n$ in case (ii) of Lemma 3.3.3.

(iv) \mathfrak{B} has an induced local boc $\mathfrak{B}_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}$ for some $l < n$ in case (ii) of Lemma 3.3.4.

The proof of Classification 3.3.2 1) Suppose a two-point wild boc is med, if \mathfrak{B}_X or \mathfrak{B}_Y is in case (iii) or (iv) of Classification 3.3.5, the induced local boc given by deleting Y or X may be considered, which is wild type. Therefore it is assumed that one of boc \mathfrak{B}_X or \mathfrak{B}_Y has Formula (3.3-5) and another is in case (i) of Lemma 3.3.3, or both of \mathfrak{B}_X and \mathfrak{B}_Y are in case (i) of Lemma 3.3.3. MW1 or MW2 follows.

2) If a local wild boc in case (iii) of Classification 3.3.5 is med, then there is an induced boc in case (ii) of Lemma 3.3.3. MW3 is reached.

3) If a local wild boc in case (iv) of Classification 3.3.5 is med, then there is an induced boc in case (ii)' of Lemma 3.3.4. MW4 or MW5 is reached. \square

3.4 Non-homogeneity in the cases of MW1-4

Throughout the subsection, $(\mathfrak{A}^0, \mathfrak{B}^0)$ denotes any pair of matrix bimodule problem and its associated boc.

Proposition 3.4.1 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW1, then \mathfrak{B}^0 is non-homogeneous.

Proof 1) Let \mathfrak{B}_X be an induced local boc of \mathfrak{B} . Suppose $\vartheta_1 : R(\mathfrak{B}_X) \rightarrow R(\mathfrak{B})$, $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ are two induced functors, and $\vartheta = \vartheta_2 \vartheta_1 : R(\mathfrak{B}_X) \rightarrow R(\mathfrak{B}^0)$. For any $\lambda \in k$, $\phi(\lambda) \neq 0$, a representation $S'_\lambda \in R(\mathfrak{B}_X)$ given by $(S'_\lambda)_X = k$, $S'_\lambda(x) = (\lambda)$ is defined. If \mathfrak{B}^0 is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq k \setminus \{\lambda \mid \phi(\lambda) \neq 0\}$, such that $\{\vartheta(S'_\lambda) \mid \lambda \in \mathcal{L}\}$ is a family of homogeneous iso-classes of $R(\mathfrak{B}^0)$. By Corollary 3.2.4, there is an almost split conflation $(e'_\lambda) : S'_\lambda \xrightarrow{\iota'} E'_\lambda \xrightarrow{\pi'} S'_\lambda$ in $R(\mathfrak{B}_X)$ with $E'_\lambda(x) = J_2(\lambda)$, such that $\vartheta(e'_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$. Fix any $\lambda \in \mathcal{L}$, and the conflation $(e_\lambda) = \vartheta_1(e'_\lambda) : S_\lambda \xrightarrow{\iota} E_\lambda \xrightarrow{\pi} S_\lambda$ in $R(\mathfrak{B})$, then $(S_\lambda)_X = k$, $(S_\lambda)_Y = 0$, $S_\lambda(x) = (\lambda)$, $S_\lambda(a_i) = 0$ for all solid arrow a_i of \mathfrak{B} , and

$(E_\lambda)_X = k^2, (E_\lambda)_Y = 0, E_\lambda(x) = J_2(\lambda), E_\lambda(a_i) = 0$. Since $\vartheta_2(e_\lambda) = \vartheta(e'_\lambda)$ is almost split in $R(\mathfrak{B}^0)$, so is (e_λ) in $R(\mathfrak{B})$ by Lemma 3.2.2 (ii) inductively.

2) Let $L \in R(\mathfrak{B})$ be an object given by $L_X = L_Y = k$, $L(x) = (\lambda)$, $L(a_1) = (1)$ and $L(a_i) = 0$ for $i > 1$. Let $g : L \rightarrow S_\lambda$ be a morphism with $g_X = (1)$, $g_Y = (0)$ and $g(v) = 0$ for all dotted arrows v of \mathfrak{B} . It is asserted that g is not a retraction. Otherwise, if there is a morphism $h : S_\lambda \rightarrow L$ such that $hg = id_{S_\lambda}$, then $h_X = (1)$ and $h_Y = (0)$. But h being a morphism implies that $(1)(1) = h_X L(a) = S_\lambda(a) h_Y = (0)(0)$, a contradiction.

3) There exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. If $\tilde{g}_X = (a, b)$, then $\tilde{g}_X \pi_X = g_X$, i.e., $(a, b) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (1), a = 1$. But \tilde{g} being a morphism implies: $\tilde{g}_X E_\lambda(x) = L(x) \tilde{g}_X$, i.e., $(1, b) \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = (\lambda)(1, b)$, $(\lambda, 1 + b\lambda) = (\lambda, \lambda b)$, a contradiction. Thus \mathfrak{B}^0 is not homogeneous. \square

Proposition 3.4.2 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW2, then \mathfrak{B}^0 is non-homogeneous.

Proof Since $f(x, y)$ is non-invertible in $k[x, y, \phi_X(x)^{-1} \phi_Y(y)^{-1}]$, after dividing the dotted arrows v_j by some powers of $\phi_X(x)$ and $\phi_Y(y)$, it may be assumed that $f(x, y) \in k[x, y]$. There exist three cases on $f(x, y) = \alpha(x)h(x, y)\beta(y)$ as in Formula (3.3-1).

Case ① $h(x, y) \notin k^*$, then $h(x, y)$ and $\phi_X(x)\phi_Y(y)$ are coprime. There is an infinite set

$$\mathcal{L}' = \{(\lambda, \mu) \in k \times k \mid h(\lambda, \mu) = 0, \phi_X(\lambda)\phi_Y(\mu) \neq 0\}.$$

by Bezout's theorem. Clearly, $\mathcal{L}_X = \{\lambda \in k \mid (\lambda, \mu) \in \mathcal{L}'\}$ is an infinite set.

Case ② $h(x, y) \in k^*$, but there is an irreducible factor $\beta'(y)$ of $\beta(y)$ coprime to $\phi_Y(y)$. If $\beta'(\mu) = 0$, then there is an infinite set $\mathcal{L}_X = \{\lambda \in k \mid \phi_X(\lambda)\phi_Y(\mu) \neq 0, \beta(\mu) = 0\}$.

Case ③ $h(x, y) \in k^*$, and $\beta(y) \mid \phi(y)^e$ for some $e \in \mathbb{Z}^+$, then there must exist an irreducible factor $\alpha'(x)$ of $\alpha(x)$ coprime to $\phi_X(x)$. If $\alpha'(\lambda) = 0$, there is an infinite set $\mathcal{L}_Y = \{\mu \in k \mid \phi_X(\lambda)\phi_Y(\mu) \neq 0, \alpha(\lambda) = 0\}$.

The cases ①–② are dealt with first in the following statement 1)–3).

1) The discussion is carried out as in proof 1) of Proposition 3.4.1, then an infinite set $\mathcal{L} \subseteq \mathcal{L}_X$ is obtained.

2) Let $L \in R(\mathfrak{B})$ be an object given by $L_X = k = L_Y$, $L(x) = (\lambda), \lambda \in \mathcal{L}; L(y) = (\mu)$ with $(\lambda, \mu) \in \mathcal{L}'$; $L(a_1) = (1); L(a_i) = 0$ for $i > 1$. Let $g : L \rightarrow S_\lambda$ be a morphism in $R(\mathfrak{B})$ with $g_X = (1)$, $g_Y = (0)$ and $g(v) = 0$ for all dotted arrows v , then g is not a retraction. Otherwise, if there is a morphism $h : S_\lambda \rightarrow L$ such that $hg = id_{S_\lambda}$, then $h_X = (1)$ and $h_Y = (0)$. But h being a morphism implies that $-1 = S_\lambda(a_1)h_Y - h_X L(a_1) = h(\delta(a_1)) = f(\lambda, \mu)h(v) = 0$, a contradiction.

3) There exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. A contradiction appears as the same as in the proof 3) of Proposition 3.4.1.

If case ③ appears, then a set of homogeneous iso-classes $\{S_\mu \mid \mu \in \mathcal{L}\}$ is used. Let L be the same as in 2), and a morphism $g : S_\mu \rightarrow L$ be not a section. There is an extension $\tilde{g} : E_\mu \rightarrow L$, which leads to a contradiction. \square

Proposition 3.4.3 [B1] If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW3 with $R = k[x, \phi(x)^{-1}]$, then \mathfrak{B}^0 is non-homogeneous.

Proof After a series of regularizations $a_i \mapsto \emptyset, i = 1, \dots, s-1$, it may be assumed that $s = 1$ in MW3. Since $f_{11}(x, \bar{x})$ is non-invertible in $k[x, \bar{x}, \phi_X(x)^{-1} \phi_X(\bar{x})^{-1}]$, by a similar discussion as the beginning of the proof of Proposition 3.4.2, there are infinite sets:

$$\begin{aligned} \mathcal{L}_1 &= \{\lambda \mid h_{11}(\lambda, \mu) = 0, \phi(\lambda) \neq 0\}; \\ \mathcal{L}_2 &= \{\lambda \mid \beta_{11}(\mu) = 0, \phi(\lambda)\beta_{11}(\lambda) \neq 0\}; \mathcal{L}_3 = \{\mu \mid \alpha_{11}(\lambda) = 0, \phi(\mu)\alpha_{11}(\mu) \neq 0\} \end{aligned}$$

according to the cases ①–③ respectively. It is easy to see that $\lambda \neq \mu$ in \mathcal{L}_1 – \mathcal{L}_3 .

Define a polynomial $\psi(x) = \phi(x) \prod_{i=1}^n \alpha_i(x) \beta_i(x)$, there is an induced boc \mathfrak{B}' of \mathfrak{B} with $R' = k[x, \psi(x)^{-1}]$ given by a localization. Note that \mathfrak{B}' is not necessarily minimal. Set the induced functors $\vartheta_1 : R(\mathfrak{B}') \rightarrow R(\mathfrak{B})$, $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$, and $\vartheta = \vartheta_2 \vartheta_1 : R(\mathfrak{B}') \rightarrow R(\mathfrak{B}^0)$. The case of \mathcal{L}_1 or \mathcal{L}_2 is dealt with first in the following 1)–3).

1) Let $S_\lambda \in R(\mathfrak{B}')$, $\lambda \in \mathcal{L}_1$ or \mathcal{L}_2 be an object given by $(S_\lambda)_X = k$, $S_\lambda(x) = (\lambda)$, $\psi(\lambda) \neq 0$, then $S_\lambda(a_i) = (0)$ for $1 \leq i \leq n$ by Lemma 3.2.6 (ii), since $f_{ii}(\lambda, \lambda) \neq 0$. If \mathfrak{B}^0 is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}_1$ or \mathcal{L}_2 , such that $\{\vartheta(S_\lambda) \in R(\mathfrak{B}^0) \mid \lambda \in \mathcal{L}\}$ is a family of homogeneous iso-classes of \mathfrak{B}^0 . By Theorem 3.2.3 with respect to x of \mathfrak{B}' , there is an almost split conflation $(e'_\lambda) : S_\lambda \xrightarrow{\iota} E_\lambda \xrightarrow{\pi} S_\lambda$ in $R(\mathfrak{B}')$, such that $\vartheta(e'_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$. By Lemma 3.1.3 (ii) it may be assumed that $\iota = (0 \ 1)$, $\pi = (1 \ 0)^T$, thus $E_\lambda(x) = \begin{pmatrix} \lambda & c \\ 0 & \lambda \end{pmatrix}$. By Lemma 3.2.6 (ii) once again, $E_\lambda(a_i) = (0)$ for $i = 1, \dots, n$, therefore $E_\lambda(x) = J_2(\lambda)$. In fact, if $E_\lambda(x)$ was λI_2 , (e'_λ) would be splittable. Fix any $\lambda \in \mathcal{L}$, and consider the conflation $(e_\lambda) = \vartheta_1(e'_\lambda)$ in $R(\mathfrak{B})$, which is almost split by Lemma 3.2.2 (ii).

2) Let $\mu \in k$ with $\phi(\mu) \neq 0$, $h_{11}(\lambda, \mu) = 0$ or $\beta_{11}(\mu) = 0$. Define $L \in R(\mathfrak{B})$ with $L_X = k^2$, $L(x) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $L(a_1) = J_2(0)$, and $L(a_i) = (0)$ for $2 \leq i \leq n$. L is well defined. In fact, if $\eta = \{\eta_X, \eta(v_j)\} : L \mapsto L$ is an isomorphism, then $L(x)\eta_X = \eta_X L(x)$ forces $\eta_X = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and $L(a_1)\eta_X - \eta_X L(a_1) = f_{11}(L(x), L(x))\eta(v_1)$ implies $\begin{pmatrix} 0 & b-a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_{11}(\lambda, \lambda)v_{11} & f_{11}(\lambda, \lambda)v_{12} \\ f_{11}(\mu, \mu)v_{21} & f_{11}(\mu, \mu)v_{22} \end{pmatrix}$, which has a solution $v_{ij} = 0, a = b$. Let $g : L \rightarrow S_\lambda$ be a morphism with $g_X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $g(v_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all j , then g is not a retraction. Otherwise, there is a morphism $h : S_\lambda \rightarrow L$ with $hg = id_{S_\lambda}$:

$$\begin{array}{ccc} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} & & \begin{pmatrix} \lambda \end{pmatrix} \\ \downarrow & \xrightarrow{g} & \downarrow \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \xrightarrow{h} & \begin{pmatrix} 0 \end{pmatrix} \end{array}$$

Thus $h_X = (1, b)$. Set $h(v_1) = (c, d)$. Then

$$S_\lambda(a_1)h_X - h_X L(a_1) = f_{11}(S_\lambda(x), L(x))h(v_1) \implies \\ 0(1, b) - (1, b) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f_{11}(\lambda, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix})(c, d) = (f_{11}(\lambda, \lambda)c, f_{11}(\lambda, \mu)d),$$

which leads to $-(0, 1) = (*, 0)$, a contradiction.

3) There is a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$. Set $\tilde{g}_X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\tilde{g}_X \pi_X = g_X$ yields $\tilde{g}_X = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. On the other hand, $\tilde{g} : L \mapsto E_\lambda$ being a morphism leads to $\tilde{g}_X E_\lambda(x) = L(x)\tilde{g}_X$, i.e., $\begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$, then $\begin{pmatrix} \lambda & 1+\lambda b \\ 0 & \lambda d \end{pmatrix} = \begin{pmatrix} \lambda & \lambda b \\ 0 & \mu d \end{pmatrix}$, a contradiction. Therefore \mathfrak{B}^0 is not homogeneous.

If \mathcal{L}_3 appears, then a set of homogeneous iso-classes $\{S_\mu \mid \mu \in \mathcal{L}\}$ is used. Let L be the same as in 2), and a morphism $g : S_\mu \rightarrow L$ be not a section. There is an extension $\tilde{g} : E_\mu \rightarrow L$, which leads to a contradiction. \square

Proposition 3.4.4 If \mathfrak{B}^0 has an induced boc \mathfrak{B} in the case of MW4 with two polynomials $\phi(x)$ and $\psi(x, x_1)$, then \mathfrak{B}^0 is non-homogeneous.

Proof Fix some $\lambda \in k$ with $\psi(\lambda, x_1) \neq 0$, $\mathcal{L}' = \{\mu \mid \psi(\lambda, \mu) \neq 0\} \subseteq k$ is a co-finite subset.

Since $\phi(\lambda) \neq 0$, a series of regularizations $a_i \mapsto \emptyset, w_i = 0$ for $i = 1, \dots, n_1 - 1$ and a loop mutation $a_{n_1} \mapsto (x_1)$ can be made in Formula (3.3-3). Thus an induced boc \mathfrak{B}' of \mathfrak{B} is obtained. Furthermore, since $f_{ii}(\lambda, x_1, \bar{x}_1)$ are invertible in $k[x_1, \psi(\lambda, x_1)^{-1}\psi(\lambda, \bar{x}_1)^{-1}]$ for $n_1 < i \leq n$, after regularizations $a_i \mapsto \emptyset, w_i = 0$ in Formula (3.3-4), an induced minimal local boc \mathfrak{B}_λ is obtained and an induced functor ϑ_1 as well.

$$\mathfrak{B}_\lambda : x_1 \bigcirc X, \quad R_\lambda = k[x_1, \psi(\lambda, x_1)^{-1}], \quad \vartheta_1 : R(\mathfrak{B}_\lambda) \rightarrow R(\mathfrak{B}).$$

Set $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$ and $\vartheta = \vartheta_2 \vartheta_1 : R(\mathfrak{B}_\lambda) \rightarrow R(\mathfrak{B}^0)$.

1) Let $S'_\mu \in R(\mathfrak{B}_\lambda)$ for any $\mu \in \mathcal{L}'$ be given by $(S'_\mu)_X = k$ and $S'_\mu(x_1) = (\mu)$. If \mathfrak{B}^0 is homogeneous, then there exists a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$, such that $\{\vartheta(S'_\mu) \in R(\mathfrak{B}^0) \mid \mu \in \mathcal{L}\}$ is a family of homogeneous iso-classes. By Corollary 3.2.4, there is an almost split conflation (e'_μ) in $R(\mathfrak{B}_\lambda)$, such that $\vartheta(e'_\mu)$ is almost split in $R(\mathfrak{B}^0)$. Fix any $\mu \in \mathcal{L}$, and consider the conflation $(e_\mu) = \vartheta_1(e'_\mu) : S_\mu \xrightarrow{\iota} E_\mu \xrightarrow{\pi} S_\mu$ in $R(\mathfrak{B})$, where $(E_\mu)_X = k^2$, $E_\mu(x) = \lambda I_2$, $E_\mu(a_{n_1}) = J_2(\mu)$, $E_\mu(a_i) = (0)$, $i \neq n_1$. Then (e_μ) is almost split by Lemma 3.2.2 (ii).

2) Define a representation L of \mathfrak{B} given by $L_X = k^2$, $L(x) = J_2(\lambda)$, $L(a_{n_1}) = \mu I_2$ and $L(a_i) = 0$ for $i \neq n_1$. L is well defined, in fact if we make an unraveling for $x \mapsto J_2(\lambda)$, then by Lemma 3.2.6 (ii), after a sequence of regularizations $\begin{pmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{pmatrix}$ (splitting from a_i) $\mapsto \emptyset$; and $\begin{pmatrix} w_{i11} & w_{i12} \\ w_{i21} & w_{i22} \end{pmatrix}$ (splitting from w_i) $= 0$ for $i = 1, \dots, n_1 - 1$, an induced boc with $\delta^0\left(\begin{pmatrix} a_{n_1,11} & a_{n_1,12} \\ a_{n_1,21} & a_{n_1,22} \end{pmatrix}\right) = 0$ is obtained. Let $g : L \rightarrow S_\mu$ be a morphism with $g_X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $g(v_j) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all possible j . It is obvious that g is not a retraction.

3) There exists a lifting $\tilde{g} : L \rightarrow E_\lambda$ with $\tilde{g}\pi = g$:

$$\begin{array}{ccc} \begin{array}{c} J_2(\lambda) \\ \curvearrowright \\ \mu I_2 \end{array} & \xrightarrow{\tilde{g}} & \begin{array}{c} \lambda I_2 \\ \curvearrowright \\ J_2(\mu) \end{array} \\ L & & E_\lambda \end{array}$$

Since $\tilde{g}_X \pi_X = g_X$, $\tilde{g}_X = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}$. On the other hand, $(x - y)^2 \mid f_{11}(x, y)$ for $\bar{w} \neq 0$ and $(J_2(\lambda) - \lambda I_2)^2 = 0$, hence $f_{11}(J_2(\lambda), \lambda I_2) = 0$. $\tilde{g} : L \rightarrow E_\lambda$ being a morphism implies that

$$\begin{aligned} L(a_1)\tilde{g}_X - \tilde{g}_X E_\lambda(a_1) &= f_{11}(L(x), E_\lambda(x))\tilde{g}(\bar{w}) = 0 \text{ for } \bar{w} \neq 0, \text{ or } 0 \text{ for } \bar{w} = 0 \\ \implies \mu \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} - \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} J_2(\mu) &= 0, \quad \text{i.e. } - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

The contradiction shows that \mathfrak{B}^0 is non-homogeneous. \square

Proposition 3.4.5 Let the boc \mathfrak{B}^0 have induced bocses $\mathfrak{B}^s, \mathfrak{B}^\varepsilon, \mathfrak{B}^\tau$ given in Formula (3.2-3) and satisfying the condition (i)–(iii) stated between Corollary 3.2.4 and 3.2.5. Then \mathfrak{B}^0 is non-homogeneous.

Proof Suppose $\mathfrak{B} = \mathfrak{B}^\varepsilon$, where $\mathcal{T} = \{X, Y\}$, $R_X = k[x, \phi(x)^{-1}]$, $R_Y = k1_Y$, the layer $L = (R; \omega; a_1, \dots, a_n; v_1, \dots, v_m)$ and $a_1 : X \rightarrow Y, \delta(a_1) = 0$:

$$\mathfrak{B} : \quad x \curvearrowright X \xrightarrow{a_1} Y \quad .$$

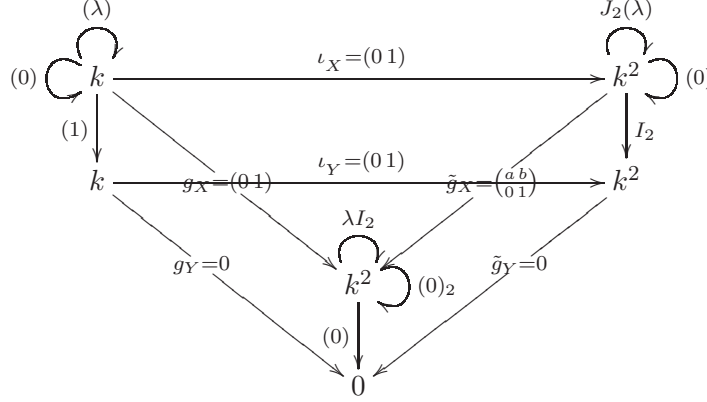
Making a reduction given by $a_1 \mapsto (1)$ of Proposition 2.2.7, an induced local boc $\mathfrak{B}' = \mathfrak{B}^\tau$ is obtained by a sequence of regularizations with $\mathcal{T}' = \{Z\}$, $R' = k[z, \phi'(z)^{-1}]$. Set the induced functors $\vartheta_1 : R(\mathfrak{B}') \rightarrow R(\mathfrak{B})$, $\vartheta_2 : R(\mathfrak{B}) \rightarrow R(\mathfrak{B}^0)$, and $\vartheta = \vartheta_2 \vartheta_1 : R(\mathfrak{B}') \rightarrow R(\mathfrak{B}^0)$.

1) For any $\lambda \in k, \phi'(\lambda) \neq 0$, there is an object $S'_\lambda \in R(\mathfrak{B}')$ with $(S'_\lambda)_Z = k, S'_\lambda(z) = (\lambda)$. If \mathfrak{B}^0 is homogeneous, then there exists a co-finite subset $\mathcal{L} \subseteq k \setminus \{\mu \mid \phi'(\mu) = 0\}$, such that $\{\vartheta(S'_\lambda) \in R(\mathfrak{B}^0) \mid \lambda \in \mathcal{L}\}$ is a family of homogeneous iso-classes. Fix any $\lambda \in \mathcal{L}$, there is an almost split conflation $(e'_\lambda) : S'_\lambda \xrightarrow{\iota} E'_\lambda \xrightarrow{\pi} S'_\lambda$ in $R(\mathfrak{B}')$ with $E'_\lambda(z) = J_2(\lambda)$, such that $\vartheta(e'_\lambda)$ is an almost split conflation in $R(\mathfrak{B}^0)$ by Corollary 3.2.5. Let $(e_\lambda) = \vartheta_1(e'_\lambda) : S_\lambda \rightarrow E_\lambda \rightarrow S_\lambda$ is an almost split conflation in $R(\mathfrak{B})$ by lemma 3.2.2 (ii) inductively, where

$$S_\lambda : (\lambda) \curvearrowright k \xrightarrow{(1)} k, \quad E_\lambda : J_2(\lambda) \curvearrowright k^2 \xrightarrow{I_2} k^2 .$$

2) Define an object $L \in R(\mathfrak{B})$ with $L_X = k^2, L_Y = 0$, $L(x) = \lambda I_2$, $L(a_i) = (0), 1 \leq i \leq n$. Define a morphism $g : S_\lambda \rightarrow L$ with $g_X = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $g_Y = 0$, and $g(v) = 0$ for any dotted arrow v . It is claimed that g is not a retraction. Otherwise, if there is a morphism $h : L \rightarrow S_\lambda$ with

$gh = id_{S_\lambda}$, then $h_X = \binom{c}{1}$. Since $h_X S_\lambda(a_1) = L(a_1)h_Y$, there is $\binom{c}{1}(1) = 0$, a contradiction.



3) There exists an extension $\tilde{g} : E_\lambda \rightarrow L$ with $\iota \tilde{g} = g$, which implies $\tilde{g}_X = \binom{a}{0} \binom{b}{1}$. Since \tilde{g} is a morphism, there is $E(x)\tilde{g}_X = \tilde{g}_X L(x)$, i.e. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, a contradiction. Therefore \mathfrak{B}^0 is not homogeneous. \square

Remark 3.4.6 For the sake of convenience, the following notation on MW5 is used. Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with R being trivial, and $(\mathfrak{A}^\nu, \mathfrak{B}^\nu)$ be an induced pair of $(\mathfrak{A}, \mathfrak{B})$ given by a sequence of reductions in the sense of Lemma 2.3.2. Suppose $(\mathfrak{A}^\nu, \mathfrak{B}^\nu)$ is local, and by calculating a series of triangular formulae according to Subsection 3.3, an induced pair

$$(\mathfrak{A}^\nu_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}, \mathfrak{B}^\nu_{(\lambda_0, \lambda_1, \dots, \lambda_{l-1})}) \quad (3.4-2)$$

is obtained. Suppose in addition, the pair (3.4-2) is in the case of MW5, which satisfies Formulae (3.3-3)–(3.3-4) with the polynomials $\phi(x)$ and $\psi(x, x_1)$ given by Formulae (3.3-8) and (3.3-9).

Denote the pair (3.4-2) by $(\mathfrak{A}^{s+1}, \mathfrak{B}^{s+1})$. It means that $(\mathfrak{A}^s, \mathfrak{B}^s)$, with the first arrow a_1^s and $\delta(a_1^s) = 0$, is obtained by a sequence of reductions in the sense of Lemma 2.3.2 starting from $(\mathfrak{A}, \mathfrak{B})$. After making a loop mutation $a_1^s \mapsto (x)$, there is an induced pair $(\mathfrak{A}^{s+1}, \mathfrak{B}^{s+1})$ with $R^{s+1} = k[x]$. Removing, by regularizations, the pairs of arrows $(a_1, w_1); \dots; (a_{n_1-1}, w_{n_1-1})$ from \mathfrak{B}^{s+1} in Formula (3.3-3), an induced pair say $(\mathfrak{A}^t, \mathfrak{B}^t)$ under the assumption $x \mapsto (\lambda)$ is obtained. Denote the solid arrows of \mathfrak{B}^t by $a_{j+1}^t, j = 0, \dots, n - n_1$, then a_1^t obtained from a_{n_1} , is the first arrow of \mathfrak{B}^t . The boc \mathfrak{B}^t is also said to be *in the case of MW5*.

4. One-sided pairs

In this section, some special quotient problems of matrix bimodule problems are defined. The purpose is to deal with the most difficult part of the proof of the main theorem in the subsections 5.4–5.5.

4.1 Definition of one-sided pairs

We give the definition of one-sided pairs; then consider the induced pairs after some reductions via pseudo formal equations in this subsection.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a matrix bimodule problem with \mathcal{T} being trivial, let $\mathfrak{C}, \mathfrak{B}$ be the associated bi-comodule problem and boc \mathfrak{B} respectively. Suppose a sequence of pairs

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^r, \mathfrak{B}^r)$$

is given by reductions in the sense of Lemma 2.3.2. Assume that the leading position of the first base matrix A_1^r of \mathcal{M}_1^r is (p^r, q^r) contained in the (\mathbf{p}, \mathbf{q}) -th leading block of A_1 of \mathcal{M} partitioned under \mathcal{T} . It is further assumed that d_1, \dots, d_m are the first m solid arrows of \mathfrak{B}^r , which locate at the p^r -th row of the formal product Θ^r , such that d_m is sitting at the last column of the (\mathbf{p}, \mathbf{q}) -block, see the picture below:

$$\begin{array}{|c|c|c|} \hline & & \\ \hline d_1 & \cdots & d_m \\ \hline \end{array} \quad (4.1-1)$$

Recalling the statement between Formula (1.2-3) and (1.2-4), the quotient problem $(\mathfrak{A}^r)^{[m]} = (R^r, \mathcal{K}^r, (\mathcal{M}^r)^{[m]}, H^r)$ of \mathfrak{A}^r ; the sub-bi-comodule problem $(\mathfrak{C}^r)^{(m)} = (R^r, \mathcal{C}^r, (\mathcal{N}^r)^{(m)}, \partial|_{(\mathcal{N}^r)^{(m)}})$ of \mathfrak{C}^r with the quasi-basis d_1, \dots, d_m of $(\mathcal{N}^r)^{(m)}$, and the sub-bocs $(\mathfrak{B}^r)^{(m)}$ of \mathfrak{B}^r , a *quotient-sub-pair* $((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$ is obtained, and it is denoted by $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ for simplicity.

Denote a set of integers by $\bar{T} = \bar{T}_R \times \bar{T}_C \subseteq T^r$, where $\bar{T}_R = \{0\}$ and $\bar{T}_C = \{1, 2, \dots, m\}$ are the row and column indices of (d_1, d_2, \dots, d_m) respectively. A representation of size vector $\underline{n} = (n_0; n_1, \dots, n_m)$ over $\bar{\mathfrak{A}}$ can be written as $\bar{M} = (\bar{M}_1, \dots, \bar{M}_m)$, where \bar{M}_i is an $n_0 \times n_i$ -matrix over k . But a morphism between two representations must be discussed returning back to the category $R(\mathfrak{A}^r)$. Moreover, if $\bar{\mathfrak{A}}'$ is any induced pair of $\bar{\mathfrak{A}}$, a pseudo functor \bar{v} can be considered acting on the objects over $\bar{\mathfrak{A}}'$. Recall the formal equation of the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$:

$$\begin{aligned} & (\sum_{Y \in \mathcal{T}^r} e_Y^r * E_Y^r) (\sum_i a_i^r * A_i^r) = (H^r (\sum_j v_j^r * V_j^r) - (\sum_j v_j^r * V_j^r) H^r) \\ & + (\sum_i a_i^r * A_i^r) (\sum_{Y \in \mathcal{T}^r} e_Y^r * E_Y^r + \sum_j v_j^r * V_j^r) - (\sum_j v_j^r * V_j^r) (\sum_i a_i^r * A_i^r). \end{aligned}$$

Let $d_i : X \rightarrow Y_i$ (possibly $Y_i = X$, or $Y_i = Y_j$ for $i \neq j$). Then the $(p^r, q^r), \dots, (p^r, q^r + m - 1)$ -th equations of the formal equation of $(\mathfrak{A}^r, \mathfrak{B}^r)$ can be rewritten as:

$$e_X(d_1, d_2, \dots, d_m) = (w_1, w_2, \dots, w_m) + (d_1, d_2, \dots, d_m) \begin{pmatrix} e_{Y_1} & w_{12} & \cdots & w_{1m} \\ & e_{Y_2} & \cdots & w_{2m} \\ & & \ddots & \vdots \\ & & & e_{Y_m} \end{pmatrix}. \quad (4.1-2)$$

Remark 4.1.1 We give some explanation on the notations of Formula (4.1-2).

(i) e_X is the (p^r, p^r) -th entry of the formal product $\sum_{Y \in \mathcal{T}^r} e_Y^r * E_Y^r$; and e_{Y_ξ} the $(q^r + \xi - 1, q^r + \xi - 1)$ -th entry of that for $\xi = 1, \dots, m$.

(ii) For $\xi = 1, \dots, m$, w_ξ is the $(p^r, q^r + \xi - 1)$ -th entry of $H^r (\sum_j v_j^r * V_j^r) - (\sum_j v_j^r * V_j^r) H^r$. In fact, $w_\xi = \sum_j \alpha_\xi^j v_j^r$, where $s(v_j^r) \ni p^r, t(v_j^r) \ni q^r + \xi - 1, \alpha_\xi^j \in k$.

(iii) For $1 \leq \eta < \xi \leq m$, $w_{\eta\xi}$ is the $(q^r + \eta - 1, q^r + \xi - 1)$ -th entry of $\sum_j v_j^r * V_j^r$. In fact, $w_{\eta\xi} = \sum_j \beta_{\eta\xi}^j v_j^r$ where $s(v_j^r) \ni q^r + \eta - 1, t(v_j^r) \ni q^r + \xi - 1, \beta_{\eta\xi}^j \in k$.

The differential of d_1, \dots, d_m can be read off from Formula (4.1-2). Note that in each monomial of the differential, there exists at most one solid arrow multiplying the dotted one from the left:

$$-\delta(d_i) = w_i + \sum_{j < i} d_j w_{ij}, \quad 1 \leq i \leq m. \quad (4.1-3)$$

Definition 4.1.2 A boc \mathfrak{B} with a layer $L = (R; \omega; a_1, \dots, a_m, b_1, \dots, b_n; \underline{u}_j, \underline{v}_j, \bar{u}_j, \bar{v}_j)$ is said to be *one-sided*, provided that R is trivial and $\mathcal{T} = \{X, Y_1, \dots, Y_h\}$; the solid arrows

$a : X \rightarrow Y, b : X \rightarrow X$, and the dotted arrows are divided into four classes: $\bar{u} : X \rightarrow X$, $\underline{u} : Y \rightarrow Y', \bar{v} : Y \rightarrow X, \underline{v} : X \rightarrow Y$; the differentials of the solid arrows are given by

$$\begin{aligned}\delta(a_i) &= \sum_j \alpha_{ij} \underline{v}_j + \sum_{i' < i, j} \beta_{ii'j} a_{i'} \underline{u}_j + \sum_{b_{i'} \prec a_i, j} \gamma_{ii'j} b_{i'} \underline{v}_j, \\ \delta(b_i) &= \sum_j \lambda_{ij} \bar{u}_j + \sum_{a_{i'} \prec b_i, j} \mu_{ii'j} a_{i'} \bar{v}_j + \sum_{i' < i, j} \nu_{ii'j} b_{i'} \bar{u}_j,\end{aligned}$$

with the coefficients $\alpha_{ij}, \beta_{ii'j}, \gamma_{ii'j}, \lambda_{ij}, \mu_{ii'j}, \nu_{ii'j} \in k$.

The associated boc \mathfrak{B} of \mathfrak{A} is one-sided by Formula (4.1-3), and $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ is called a *one-sided pair*.

$$(4.1-4)$$

Let $(\mathfrak{A}, \mathfrak{B})$ be any pair, $(\mathfrak{A}^{[h]}, \mathfrak{B}^{(h)})$, $h \geq 1$, be a quotient-sub-pair. Note that if the reduction on $(\mathfrak{A}, \mathfrak{B})$ is made with respect to an admissible $R' - \bar{R}$ -bimodule L by Proposition 2.2.1–2.2.7, then L is also the admissible bimodule of $(\mathfrak{A}^{[h]}, \mathfrak{B}^{(h)})$. Thus there are two sequences of reductions as below, such that $(\bar{\mathfrak{A}}^{i+1}, \bar{\mathfrak{B}}^{i+1})$ and $(\mathfrak{A}^{r+i+1}, \mathfrak{B}^{r+i+1})$ are obtained respectively from $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ and $(\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i})$ by the same admissible bimodule, or by the same regularization 2.1.8 for $0 \leq i < s$:

$$\begin{aligned}(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}), \quad (\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1), \quad \dots, \quad (\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i) \quad \dots, \quad (\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s); \\ (\mathfrak{A}^r, \mathfrak{B}^r), \quad (\mathfrak{A}^{r+1}, \mathfrak{B}^{r+1}), \quad \dots, \quad (\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i}), \quad \dots, \quad (\mathfrak{A}^{r+s}, \mathfrak{B}^{r+s}).\end{aligned} \quad (4.1-5)$$

Let $\bar{\mathfrak{A}}^i = (\mathfrak{A}^{r+i})^{[m^i]} = (R^{r+i}, \mathcal{K}^{r+i}, \bar{\mathcal{M}}^i, F^i)$ be a quotient problem with m^i being the number of the base matrices of $\bar{\mathcal{M}}^i$, and the associated sub-bi-comodule problem $\bar{\mathfrak{C}}^i = (\mathfrak{C}^{r+i})^{(m^i)} = (R^{r+i}, \mathcal{C}^{r+i}, \bar{\mathcal{N}}^i, \bar{\mathcal{O}}^i)$. A formula of the pair $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ is written as follows for $i = 0, 1, \dots, s$:

$$e_X^i(F^i + \bar{\Theta}^i) = (W_1, \dots, W_m) + (F^i + \bar{\Theta}^i) \begin{pmatrix} e_{Y_1}^i & W_{12} & \cdots & W_{1m} \\ & e_{Y_2}^i & \cdots & W_{2m} \\ & & \ddots & \vdots \\ & & & e_{Y_m}^i \end{pmatrix} \quad (4.1-6)$$

where the diagonal parts of e_X^i and the most right matrix in Formula (4.1-6) can be viewed as a reduced formal product $\bar{\Upsilon}^i$ of $(\mathcal{K}_0, \mathcal{C}_0)$. $\bar{\Theta}^i$ is the formal product of $(\bar{\mathcal{M}}_1^i, \bar{\mathcal{N}}_1^i)$ containing the solid arrows splitting from d_1, \dots, d_m ; $F^i + \bar{\Theta}^i$ is a $(1 \times m)$ -partitioned matrix under $\bar{\mathcal{T}}$ with a size vector $\underline{n}^i = (n_0^i; n_1^i, \dots, n_m^i)$, and F^i is sitting in the blank part in the picture:

$$F^i + \bar{\Theta}^i = \begin{array}{|c|c|c|c|c|c|} \hline & & d_q^n \cdots & \cdots & \cdots & \cdots \\ \hline & & & d_p^n & d_{p+1}^n & \cdots \\ \hline & & & & d_1^n & d_2^n \cdots \\ \hline & & & & & \cdots & d_{p-1}^n \\ \hline \end{array}$$

Reductions are performed according to Theorem 2.4.4. More precisely, the system $\bar{\mathbb{F}}^{r^i}$ of Formula (2.4-5) for the pair $(\mathfrak{A}^{r+i}, \mathfrak{B}^{r+i})$ can be written as $\bar{\mathcal{F}}^i$ below, which is said to be the *reduced defining system of the pair* $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$.

$$\bar{\mathcal{F}}^i : \bar{\Psi}_{\underline{n}^i}^l F^i \equiv \prec_{(\bar{p}^i, \bar{q}^i)} \bar{\Psi}_{\underline{n}^i}^m + F^i \bar{\Psi}_{\underline{n}^i}^r, \quad (4.1-7)$$

where the upper indices l, m, r are used to show the left, middle and right parts of the variable matrix $\bar{\Psi}_{\underline{n}^i}$; $\bar{\Psi}_{\underline{n}^i}^l$ is the strict upper triangular part of e_X^i ; $\bar{\Psi}_{\underline{n}^i}^r$ is that of the most right matrix in Formula (4.1-6); and $\bar{\Psi}_{\underline{n}^i}^m = (W_1, \dots, W_m)$. Namely, $\bar{\mathcal{F}}^i$ is obtained from Formula (4.1-6) by removing the term $\bar{\Theta}^i$ and the diagonal parts of the most left and right matrices. The strict upper triangular parts of e_X^i and $e_{Y_j}^i$ for $1 \leq j \leq m$ are constructed inductively in Proof 2) of Theorem 2.4.4; while W_h and W_{hl} are the splitting of w_h, w_{hl} .

Since it is difficult to calculate the dotted arrows after a reduction, the linear relation of the dotted elements of $\bar{\Psi}$ appearing in the reduction will be described instead.

On the other hand, $\bar{\Pi}^i = \bar{\Psi}_{\underline{n}^i}$ may be said to be a *pseudo formal product* of $(\mathcal{K}_1^i, \mathcal{C}_1^i)$; and Formula (4.1-6) a *pseudo formal equation of the pair* $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$, since the entries of $\bar{\Pi}^i$ are dotted elements with some linear relations. The reason that we borrowed the concept of “formal product” was that it is possible to read off the differential of the solid arrows from Formula (4.1-6) according to Theorem 1.4.2. For example for the first arrow $d_{l\bar{p}\bar{q}}^i$:

$$-\delta(d_{l\bar{p}\bar{q}}^i) = w_{l\bar{p}\bar{q}}^i + \sum_{j < l, q} d_{j\bar{p}\bar{q}}^{i,0} w_{jl, q\bar{q}}^i + \sum_{q < \bar{q}} d_{l\bar{p}\bar{q}}^{i,0} w_{Yq\bar{q}}^i - \sum_{q > \bar{p}} w_{X\bar{p}\bar{q}}^i d_{lq\bar{q}}^{i,0}, \quad (4.1-8)$$

where $d_{l\bar{p}\bar{q}}^i$ is split from $d_l : X \rightarrow Y$, $d_{j\bar{p}\bar{q}}^{i,0}$ is the (p, q) -th entry obtained from d_j in F^i .

Remark 4.1.3 Let $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ be any induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ after several reductions in the sense of Lemma 2.3.2. And $(\bar{\mathfrak{A}}'', \bar{\mathfrak{B}}'')$ is an induced pair of $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ given by one of three reductions of 2.3.2.

- (i) If there is a linear relation of dotted elements $\sum_j u_j = 0$ in $\bar{\Psi}_{\underline{n}'}$, then $\sum_j \bar{u}_j = 0$ in $\bar{\Psi}_{\underline{n}''}$ with \bar{u}_j being the split of u_j .
- (ii) Suppose a'_1 is the first arrow of $\bar{\mathfrak{B}}'$, and $\delta(a'_1) = v + \sum_j \alpha_j u_j$, where v, u_j are dotted elements of $\bar{\Psi}_{\underline{n}'}$. If v is a dotted arrow, and $v \notin \{u_j\}$, then $\delta(a'_1) \neq 0$.
- (iii) Set $a'_1 \mapsto \emptyset$ in (ii) by a regularization, it is said that v is *replaced by* $-\sum_j u_j$ in $\bar{\Psi}_{\underline{n}''}$.
- (iv) If we are able to determine that a dotted element v of $\bar{\Psi}_{\underline{n}''}$ is linearly independent of all the others, then v is said to be a *dotted arrow preserved in* $\bar{\mathfrak{B}}''$.

4.2 Differentials in one-sided pairs

In this subsection, the classification of local one-sided bocses is discussed; and the differentials of the solid arrows of non-local bocses are calculated.

For the sake of simplicity, the black letter δ is used instead of the symbol “ $-\delta$ ”, see Formulae (4.1-3) and (4.1-8), up to the end of the whole section 4. First of all, Classification 3.3.5 gives the following Classification on local one-sided bocses. The forms in one-sided case are much simpler than those in general case.

Classification 4.2.1 Let $\bar{\mathfrak{B}}$ be a local one-sided boc with a layer (letter a is changed to the letter b):

$$L = (R; \omega; b_1, \dots, b_n; v_1, \dots, v_m).$$

- (i) $\bar{\mathfrak{B}}$ with $R = k1_X$ has differentials by Formula (3.3-5) after some base changes:

$$\delta^0(b_1) = \bar{u}_1, \delta^0(b_2) = \bar{u}_2, \dots, \delta^0(b_n) = \bar{u}_n.$$

- (ii) $\bar{\mathfrak{B}}$ with $R = k1_X$ has some integer $1 \leq n_0 < n$, such that the differentials are

$$\begin{aligned} \delta^0(b_i) &= \bar{u}_i, \quad 1 \leq i < n_0; \quad \delta^0(b_{n_0}) = \sum_{i=1}^{n_0-1} f_{n_0 i} \bar{u}_i, \quad f_{n_0 i} \in k; \\ \delta^1(b_i) &= \sum_{j \neq n_0, j=1}^{i-1} f_{ij}(b_{n_0}) \bar{u}_j + f_{ii} \bar{u}_i, \quad \text{with respect to } b_{n_0}, \\ \text{where, } n_0 < i \leq n, \quad f_{ij}(b_{n_0}) &= \beta_{i,j}^0 + \beta_{i,j}^1 b_{n_0} \in k[b_{n_0}] \text{ for } i < j, \quad f_{ii} \in k^*. \end{aligned} \quad (4.2-1)$$

$\bar{\mathfrak{B}}$ with $R = k[x]$ is given by a sequence of regularizations, followed by $a_{n_0} \mapsto x$ in Formula (3.3-6). For the sake of simplicity, Formulae (3.3-2) and (3.3-3) are written in a unified form:

$$\begin{cases} \delta^0(b_1) &= f_{11}(x)\bar{u}_1, \\ \delta^0(b_2) &= f_{21}(x)\bar{u}_1 + f_{22}(x)\bar{u}_2, \\ &\dots \dots \\ \delta^0(b_t) &= f_{t1}(x)\bar{u}_1 + f_{t2}(x)\bar{u}_2 + \dots + f_{t,t-1}(x)\bar{u}_{t-1} + f_{tt}(x)\bar{u}_t, \end{cases} \quad (4.2-2)$$

where $f_{ij}(x) = \alpha_{i,j}^0 + \alpha_{i,j}^1 x \in k[x]$ for $1 \leq i \leq j \leq t$, and $f_{ii}(x) \neq 0$ for $1 \leq i < t$.

(iii) $\bar{\mathfrak{B}}$ has an induced boc $\bar{\mathfrak{B}}_{(\lambda_0, \dots, \lambda_t)}$ with Formula (4.2-2), where $t = n$, $f_{nn}(x) \neq 0$; $\phi(x) = 1$ in Formula (3.3-7); and there is some minimal $1 \leq s \leq n$, such that $f_{ss}(x) \in k[x] \setminus k$ is non-invertible in $k[x]$. It is in the case of MW3.

(iv) $\bar{\mathfrak{B}}$ has an induced boc $\bar{\mathfrak{B}}_{(\lambda_0, \dots, \lambda_{t-1})}$ with Formula (4.2-2), where $t = n_1 < n$, $f_{n_1, n_1}(x) = 0$ and $\phi(x) = \prod_{i=1}^{n_1-1} f_{ii}(x)$ in Formula (3.3-8). Denoting b_{n_1} by x_1 , Formula (3.3-4) shows:

$$\begin{cases} \delta^1(b_{n_1+1}) = K_{n_1+1} + f_{n_1+1, n_1+1}(x)\bar{u}_{n_1+1}, \\ \delta^1(b_{n_1+2}) = K_{n_1+2} + f_{n_1+1, n_1+1}(x, x_1)\bar{u}_{n_1+1} + f_{n_1+2, n_1+2}(x)\bar{u}_{n_1+2}, \\ \dots \dots \\ \delta^1(b_n) = K_n + h_{n, n_1+1}(x, x_1)\bar{u}_{n_1+1} + \dots + f_{n, n-1}(x, x_1)\bar{u}_{n-1} + f_{nn}(x)\bar{u}_n, \end{cases} \quad (4.2-3)$$

where $f_{ij}(x, x_1) \in k[x, x_1]$ for $i < j$, $f_{ii}(x) \neq 0$; and the polynomial $\psi(x) = \phi(x) \prod_{i=n_1+1}^n c_i(x)f_{ii}(x)$ is given by Formula (3.3-9). It is in the case of MW4.

Note in particular, that MW5 never occurs in one-sided case.

Proof The one-sided boc of (i) is finite type, and that of (ii) is tame infinite.

(iii) There is no localization needed in Formula (4.2-2), thus $c_i(x) = 1$. It is clear that $h_{ii}(x, x) = 1$ in Formula (3.3-1), therefore $\phi(x) = 1$. Finally, any non-zero and non-invertible polynomial of $k[x]$ belongs to $k[x] \setminus k$.

(iv) Suppose the last terms of the formulae in (4.2-3) are $0 \neq f'_{ii}(x, x_1) = f_{ii}(x)h_{ii}(x, x_1)$ with $h_{ii}(x, x_1) \in k[x, x_1, \psi(x)^{-1}] \setminus k[x]$ or $h_{ii}(x, x_1) = 1$. If there exists a minimal integer $n_1 < s \leq n$, such that $k_{ss}(x, x_1) \notin k[x]$, then for any $\lambda \in k$ with $\psi(\lambda) \neq 0$, the induced boc $\bar{\mathfrak{B}}_\lambda$ given by $x \mapsto (\lambda)$ returns to case (iii). Therefore $f'_{ii}(x, x_1) = f_{ii}(x) \in k[x]$ for $n_1 < i \leq n$. \square

From now on, a general one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with $|\bar{\mathcal{T}}| > 1$ is dealt with. If $\bar{\mathfrak{B}}_X$, the induced local boc of $\bar{\mathfrak{B}}$, is in case (iii) or (iv) of Classification 4.2.1, then $\bar{\mathfrak{B}}_X$ is wild and non-homogeneous, so is $\bar{\mathfrak{B}}$. Since $\bar{\mathfrak{B}}_X$ in case (i) of 4.2.1 is relatively simple, the discussion below is concentrated on $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1) in case (ii) of 4.2.1. Denote the solid edges of $\bar{\mathfrak{B}}$ before b_{n_0} by a_1, \dots, a_h . The differential δ^0 acting on a 's has two possible expressions. First,

$$\delta^0(a_1) = \underline{v}_1, \dots, \delta^0(a_h) = \underline{v}_h. \quad (4.2-4)$$

Second, there exists some $1 \leq h_1 < h$, such that $\delta^0(a_1) = \underline{v}_1, \dots, \delta^0(a_{h_1-1}) = \underline{v}_{h_1-1}$, but $\delta^0(a_{h_1}) = \sum_{j=1}^{h_1-1} \bar{\alpha}_{h_1, j} \underline{v}_j$. Inductively, there are two subsets $\{h_1, \dots, h_s\} \subseteq \{1, \dots, h\}$, and $\Lambda = \{1, \dots, h\} \setminus \{h_1, \dots, h_s\}$, such that

$$\begin{cases} \delta^0(a_i) &= \underline{v}_i, & i \in \Lambda; \\ \delta^0(a_{h_l}) &= \sum_{j \in \Lambda, j < h_l} \bar{\alpha}_{h_l, j} \underline{v}_j, & l = 1, \dots, s. \end{cases} \quad (4.2-5)$$

Convention 4.2.2 Suppose $\bar{\mathfrak{B}}$ is a one-sided boc with $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1). All the loops b_1, \dots, b_n at X are called *b-class arrows*, where the loop $\bar{b} = b_{n_0}$ is said to be *effective* or *b-class*, the others are *non-effective*. The edges a_1, \dots, a_h before \bar{b} are called *a-class arrows*, where $\{\bar{a}_i = a_{h_i} \mid 1 \leq i \leq s\}$ are said to be *effective* or *a-class*, the others are *non-effective*. Let c_1, c_2, \dots, c_t be the solid edges after \bar{b} , which are called *c-class arrows*, and they are effective.

A solid arrow splitting from one of the classes $a, \bar{a}, b, \bar{b}, c$, or a dotted element splitting from a dotted arrow of $\underline{u}, \underline{v}, \bar{u}, \bar{v}$ -classes in Picture (4.1-4) are said to be in the same class.

A special case of the differential δ^1 with respect to \bar{b} on c -class arrows is given by

$$\begin{cases} \delta^1(c_1) = \sum_{j \in \Lambda} \gamma_{1j}(\bar{b}) \underline{v}_j + \gamma_{1,h+1}(\bar{b}) \underline{v}_{h+1}, \\ \dots \dots \dots \\ \delta^1(c_t) = \sum_{j \in \Lambda} \gamma_{tj}(\bar{b}) \underline{v}_j + \gamma_{t,h+1}(\bar{b}) \underline{v}_{h+1} + \dots + \gamma_{t,h+t}(\bar{b}) \underline{v}_{h+t}, \end{cases} \quad (4.2-6)$$

where $\{\underline{v}_j\}_{j \in \Lambda} \cup \{\underline{v}_{h+j}\}_{1 \leq j \leq t}$ are dotted arrows, $\gamma_{i,h+i}(\bar{b}) \neq 0$ for $1 \leq i \leq t$.

Lemma 4.2.3 Let $\bar{\mathfrak{B}}$ be a one-sided bocs with $\bar{\mathfrak{B}}_X$ being in case (ii) of Classification 4.2.1. If Formula (4.2-6) fails, i.e. there exists some minimal $1 \leq l \leq t$ with $\gamma_{l,h+l}(\bar{b}) = 0$, then $\bar{\mathfrak{B}}$ is non-homogeneous.

Proof It is proceed with a sequence of regularizations: $a_j \mapsto \emptyset, \underline{v}_j = 0$ for $j \in \Lambda$, $b_j \mapsto \emptyset, \bar{u}_j = 0$ for $1 \leq j < n_0$, and edge reductions $\bar{a}_{h_i} \mapsto (0)$ for $i = 1, \dots, s$. Then after a loop mutation $\bar{b} \mapsto (x)$, and defining a polynomial $\phi(x) = \prod_{i=1}^{l-1} \gamma_{i,h+i}(x)$ by Formula (4.2-6), an induced pair $(\mathfrak{A}', \mathfrak{B}')$ is obtained, such that $R'_X = k[x, \phi(x)^{-1}]$ and \mathfrak{B}'_X is minimal. Without loss of generality, suppose $\mathcal{T}' = \{X, Y\}$ with $R_Y = k$. Making regularizations $c_i \mapsto \emptyset, \underline{v}'_{h+i} = 0$ for $1 \leq i < l$, an induced boc $\bigcirc \cdot \xrightarrow{c_l} \cdot$ of two vertices with $\delta(c_l) = 0$ follows, which is in the case of MW1. Thus $\bar{\mathfrak{B}}'$, consequently $\bar{\mathfrak{B}}$, are wild and non-homogeneous. \square

Let $\bar{\delta}$ be obtained from the differential δ by removing all the monomial involving any non-effective a, b -class solid arrows. Now $\bar{\delta}$ acting on all a, b, c -class arrows is written in the following three formulae:

$$\begin{aligned} \bar{\delta}(a_i) &= \underline{v}_i + \sum_{h_l < i} \bar{a}_l(\sum_j \epsilon_{ilj} \underline{u}_j), \quad i \in \Lambda; \\ \bar{\delta}(\bar{a}_\tau) &= \sum_{j < h_\tau} \bar{\alpha}_{\tau j} \underline{v}_j + \sum_{l < \tau} \bar{a}_l(\sum_j \bar{\epsilon}_{\tau l j} \underline{u}_j), \quad 1 \leq \tau \leq s. \end{aligned} \quad (4.2-7)$$

$$\begin{aligned} \bar{\delta}(b_i) &= \bar{u}_i + \sum_{\bar{a}_l < b_i} \bar{a}_l(\sum_j \epsilon_{ilj} \bar{v}_j), \quad i < n_0; \\ \bar{\delta}(\bar{b}) &= \sum_{j=1}^{n_0-1} \bar{\beta}_j \bar{u}_j + \sum_{l=1}^s \bar{a}_l(\sum_j \bar{\epsilon}_{lj} \bar{v}_j), \quad i = n_0; \\ \bar{\delta}(b_i) &= \bar{u}_i + \sum_{j=1}^{i-1, j \neq n_0} \beta_{ij}(\bar{b}) \bar{u}_j + \sum_{l=1}^s \bar{a}_l(\sum_j \epsilon_{ilj} \bar{v}_j) + \sum_{c_l < b_i} c_l(\sum_j \epsilon'_{ilj} \bar{v}_j), \quad i > n_0. \end{aligned} \quad (4.2-8)$$

$$\bar{\delta}(c_\tau) = \sum_{l=1}^s \bar{a}_l(\sum_j \zeta_{\tau l j} \underline{u}_j) + \sum_{j \leq h+\tau} \gamma_{\tau j}(\bar{b}) \underline{v}_j + \sum_{l=1}^{i-1} c_l(\sum_j \xi_{\tau l j} \underline{u}_j), \quad 1 \leq \tau \leq t. \quad (4.2-9)$$

where all the coefficients $\epsilon, \bar{\epsilon}, \bar{\alpha}, \epsilon, \epsilon', \bar{\epsilon}, \bar{\beta}, \zeta, \xi \in k$, and $\beta(\bar{b}) = \beta^0 + \beta^1 \bar{b}, \gamma(\bar{b}) = \gamma^0 + \gamma^1 \bar{b} \in k[\bar{b}]$.

4.3. Reduction sequences of one-sided pairs

The purpose of this subsection is tow folds: 1) present a condition on formal products, which can be preserved after some edge reductions; 2) construct a reduction sequence based on 1) starting from a non-local one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1).

Condition 4.3.1 (BRC) Let $(\mathfrak{A}, \mathfrak{B})$ be any pair with trivial \mathcal{T} .

(i) Suppose the solid arrows $\mathcal{D} = \{d_1, \dots, d_q\}$ and $\mathcal{E} = \{e_1, \dots, e_p\}$ locate in the lowest non-zero row of Θ and form the first $p+q$ arrows of \mathfrak{B} (not necessarily fulfilling of the whole row), such that e_1, \dots, e_{p-1} are edges starting from X , $e = e_p$ is a loop at X , and $d_i < e_p, 1 \leq i \leq q$. There exists a set of dotted arrows $\mathcal{U} = \{u_1, \dots, u_q\}$, whose complement in \mathcal{V}^* is $\mathcal{W} = \{w_1, \dots, w_t\}$.

(ii) Denote by $\bar{\delta}$ the part of the differential of a solid arrow in $\mathcal{D} \cup \mathcal{E}$ by removing all the monomials containing any solid arrow in \mathcal{D} .

$$\begin{aligned} \bar{\delta}(d_i) &= u_i + \sum_{j=1}^t (\sum_{e_l < d_i} \lambda_{ijl} e_l) w_j, \quad 1 \leq i \leq q; \\ \bar{\delta}(e_i) &= \sum_{d_j < e_i} \alpha_{ij} u_j + \sum_{j=1}^t (\sum_{l=1}^{i-1} \mu_{ijl} e_l) w_j, \quad 1 \leq i \leq p, \end{aligned}$$

where the coefficients $\lambda_{ijl}, \alpha_{ij}, \mu_{ijl} \in k$. Then it is said that $(\mathfrak{A}, \mathfrak{B})$ satisfies the *bottom row condition* with respect to $(\mathcal{D}, \mathcal{U})$ and $(\mathcal{E}, \mathcal{W})$, or (BRC) for short.

From now on, we use $G(k)$ instead of the reduction block G given below Lemma 2.3.2 for the sake of simplicity up to the end of Section 4, if there is not any confusion to be caused. Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p, q > 1$, and the first arrow of \mathfrak{B} is $a_1 : X \rightarrow Y$. Now the condition (BRC) on the induced pair $(\mathfrak{A}', \mathfrak{B}')$ is discussed.

Case (i) $a_1 = d_1$, and $d_1 \mapsto \emptyset$, let $\mathcal{D}' = \{d_2, \dots, d_q\}, \mathcal{U}' = \{u_2, \dots, u_q\}$ and $\mathcal{E}' = \mathcal{E}, \mathcal{W}' = \mathcal{W}$.

Case (ii) $a_1 = e_1$, and $e_1 \mapsto (0)$, let $\mathcal{D}' = \mathcal{D}, \mathcal{U}' = \mathcal{U}; \mathcal{E}' = \{e_2, \dots, e_p\}, \mathcal{W}' = \mathcal{W}$.

Denote by \bullet_i a solid or dotted arrow of \mathfrak{B} . Suppose after a reduction below, \bullet_i splits into a 2×1 or 2×2 matrix in \mathfrak{B}' . Then the arrows at the second row of the matrix are denoted by \bullet_{i2} , or $\bullet_{i21}, \bullet_{i22}$.

Case (iii) $a_1 = e_1$, and $e_1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, let $\mathcal{D}' = \{d_{i21}, d_{i22} \mid t(d_i) = X\} \cup \{d_{j2} \mid t(d_j) \neq X\}$, $\mathcal{U}' = \{u_{i21}, u_{i22} \mid t(d_i) = X\} \cup \{u_{j2} \mid t(d_j) \neq X\}$; and $\mathcal{E}' = \{e_{22}, \dots, e_{p-1,2}, e_{p21}, e_{p22}\}$, \mathcal{W}' is obtained by the split of \mathcal{W} and an additional dotted arrow $\nu_1^*(e_{z(X,1)} \otimes_k f_{z(X,2)})$ defined in Proof 2) of Proposition 2.1.5.

Case (iv) $a_1 = e_1$ and $e_1 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, let $\mathcal{D}' = \{d_{i21}, d_{i22} \mid t(d_i) = X \text{ or } Y\} \cup \{d_{j2} \mid t(d_j) \neq X, Y\}$, $\mathcal{U}' = \{u_{i21}, u_{i22} \mid t(d_i) = X \text{ or } Y\} \cup \{u_{j2} \mid t(d_j) \neq X, Y\}$; and $\mathcal{E}' = \{e_{i21}, e_{i22} \mid t(e_i) = Y\} \cup \{e_{j2} \mid t(e_j) \neq Y\} \cup \{e_{p21}, e_{p22}\}$, \mathcal{W} is obtained by the split of \mathcal{W} and two additional dotted arrows $\nu_1^*(e_{z(X,1)} \otimes_k f_{z(X,2)}), \nu_1^*(e_{z(Y,1)} \otimes_k f_{z(Y,2)})$ given in Proof 2) of 2.1.5.

Lemma 4.3.2 Suppose the pair $(\mathfrak{A}, \mathfrak{B})$ satisfies (BRC) with $p, q > 1$, and the first arrow of \mathfrak{B} is $a_1 : X \rightarrow Y$. Then after making a reduction $a_1 \mapsto G$ as above (i)–(iv), the induced pair $(\mathfrak{A}', \mathfrak{B}')$ satisfies (BRC) with respect to $(\mathcal{D}', \mathcal{U}')$ and $(\mathcal{E}', \mathcal{W}')$.

Proof The cases (i) and (ii) are trivial.

Suppose the reduction block $G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in (iii), resp. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in (iv), then X, Y split into two vertices X', Y' , resp. three vertices X', Y', Y'' :

$$e_X \mapsto e'_X = \begin{pmatrix} e_{Y'} & w \\ & e_{X'} \end{pmatrix}, \quad e_Y \mapsto e'_Y = e_{Y'}, \quad \text{or} \quad e_Y \mapsto e'_Y = \begin{pmatrix} e_{Y''} & w' \\ & e_{Y'} \end{pmatrix}.$$

In above-mentioned two cases, by condition (i) of (BRC) on $(\mathfrak{A}, \mathfrak{B})$, e_{i2} or e_{i21}, e_{i22} for $1 < i < p$ start at X' , but do not end at X' , because $t(e_i) \neq X$; the edge $e_{p21} : X' \rightarrow Y'$, and the loop $e_{p22} : X' \rightarrow X'$. Therefore the pair $(\mathfrak{A}', \mathfrak{B}')$ still satisfies (i) of (BRC). By condition (ii) of (BRC) on $(\mathfrak{A}, \mathfrak{B})$, we have

$$\begin{aligned} \bar{\delta}(D_i) &= U_i + \sum_{j=1}^t \lambda_{ij1} G W_j + \sum_{j=1}^t (\sum_{e_l \prec d_i} \lambda_{ijl} E_l) w_j, \\ \bar{\delta}(E_i) &= \sum_{d_j \prec e_i} \alpha_{ij} U_j + \sum_{j=1}^t \mu_{ij1} G W_j + \sum_{j=1}^t (\sum_{l=2}^{i-1} \mu_{ijl} E_l) W_j + \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} E_i - (0 \text{ or } E_i \begin{pmatrix} 0 & w' \\ 0 & 0 \end{pmatrix}), \end{aligned}$$

where $\bar{\delta}(M) = (\bar{\delta}(a_{ij}))$ for $M = (a_{ij})$. Since the bottom row of G is (0) or (00) , $(\mathfrak{A}', \mathfrak{B}')$ still satisfies condition (ii) of (BRC). \square

Lemma 4.3.3 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1), $\bar{\mathcal{T}}$ being trivial and $|\bar{\mathcal{T}}| > 1$. Then $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies (BRC) with respect to the sets:

$$\begin{aligned} \mathcal{D} &= \{a_i, i \mid \Lambda\} \cup \{b_j \mid j < n_0\}, \quad \mathcal{U} = \{v_i \mid i \in \Lambda\} \cup \{\bar{u}_j \mid j < n_0\}; \\ \mathcal{E} &= \{\bar{a}_\tau, 1 \leq \tau \leq s\} \cup \{\bar{b}\}, \quad \mathcal{W} = \{v_i \mid i \notin \Lambda\} \cup \{\bar{u}_j \mid j > n_0\} \cup \{\underline{u}, \bar{v}\text{-class arrows}\}. \end{aligned}$$

Theorem 4.3.4 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1), $\bar{\mathcal{T}}$ being trivial and $|\bar{\mathcal{T}}| > 1$. Then there exists a sequence of reductions in the sense of Lemma 2.3.2 as the first part of a sequence towards a pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5 given by Remark 3.4.6:

$$(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}) = (\bar{\mathfrak{A}}^0, \bar{\mathfrak{B}}^0), (\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1), \dots, (\bar{\mathfrak{A}}^\gamma, \bar{\mathfrak{B}}^\gamma), (\bar{\mathfrak{A}}^{\gamma+1}, \bar{\mathfrak{B}}^{\gamma+1}), \dots, (\bar{\mathfrak{A}}^{\kappa-1}, \bar{\mathfrak{B}}^{\kappa-1}), (\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa) \quad (4.3-1)$$

where κ is the minimal index, such that the pair $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ satisfies the following condition (B).

Condition (B). If a row of the formal product $\bar{\Theta}^\beta$ of $(\mathcal{M}_1^\beta, \mathcal{N}_1^\beta)$ contains some \bar{b} -class arrows of $\bar{\mathfrak{B}}^\beta$, then the same row of F^β contains one and only one nonzero entry which is a link in some reduction block G_β^i of H^β obtained by an edge reduction.

- (i) For $i = 0, 1, \dots, (\kappa - 2)$, the reduction from $\bar{\mathfrak{A}}^i$ to $\bar{\mathfrak{A}}^{i+1}$ is a composition of a series of reductions $\bar{\mathfrak{A}}^i = \bar{\mathfrak{A}}^{i,0}, \bar{\mathfrak{A}}^{i,1}, \dots, \bar{\mathfrak{A}}^{i,r_i}, \bar{\mathfrak{A}}^{i,r_i+1} = \bar{\mathfrak{A}}^{i+1}$:
 - ① For $0 \leq j < r_i - 1$, the reduction from $\bar{\mathfrak{A}}^{i,j}$ to $\bar{\mathfrak{A}}^{i,j+1}$ is a sequence of regularizations for non-effective a, b -class arrows, and finally an edge reduction of the form (0) for an effective a or b -class arrow. The reduction from $\bar{\mathfrak{A}}^{i,r_i-1}$ to $\bar{\mathfrak{A}}^{i,r_i}$ is a sequence of regularizations for non-effective a, b -class arrows.
 - ② The first arrow $a_1^i : X^i \mapsto Y^i$ of $\bar{\mathfrak{B}}^{i,r_i}$ is an effective a or b -class edge with $\delta(a_1^i) = 0$. Making an edge reduction $a_1^i \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the last term $\bar{\mathfrak{A}}^{i,r_i+1} = \bar{\mathfrak{A}}^{i+1}$ is obtained.
- (ii) It is possible that there exist a minimal integer γ , and an index $1 \leq j \leq r_\gamma + 1$, such that the first arrow of $\bar{\mathfrak{B}}^{\gamma,j}$ locates outside the matrix block coming from \bar{b} under $\bar{\mathcal{T}}$, but the first arrow of $\bar{\mathfrak{B}}^{\gamma,j+1}$ locates at the first column of the block.
- (iii) The reduction from $\bar{\mathfrak{A}}^{\kappa-1,0}$ to $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}}$ is a composition of a series of reductions given by ① of (i). There are two possibilities.
 - ① The first arrow $a_1^{\kappa-1}$ of $\bar{\mathfrak{B}}^{\kappa-1,r_{\kappa-1}}$ is an effective a or b -class solid edge with $\delta(a_1^{\kappa-1}) = 0$. Making an edge reduction $a_1^{\kappa-1} \mapsto (1)$ or $(0 \ 1)$, the last term $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}+1} = \bar{\mathfrak{A}}^\kappa$ is obtained.
 - ② The first arrow $a_1^{\kappa-1}$ is an effective b -class loop at the down-right corner of the matrix block coming from \bar{b} under $\bar{\mathcal{T}}$ with $\delta(a_1^{\kappa-1}) = 0$. Making a loop reduction $a_1^{\kappa-1} \mapsto W$, a Weyr matrix over k , $\bar{\mathfrak{A}}^{\kappa-1,r_{\kappa-1}+1} = \bar{\mathfrak{A}}^\kappa$ is obtained.

Proof If there is not any \bar{a} -class edges, i.e. $s = 0$, then after a series of regularizations, the unique effective loop in the induced bocs becomes the first arrow with $\delta(\bar{b}) = 0$. Since the induced pair is not local, but the parameter x appears only in a local pair by Remark 3.4.6, set $\bar{b} \mapsto W$ by a loop reduction of Lemma 2.3.2, the final pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$ satisfies ② of (iii) with $\kappa = 1$. Suppose $s > 0$, regularizations are made on a_i, b_j before \bar{a}_1 , the corresponding $\underline{v}_i, \bar{u}_j = 0$. Thus $\delta(\bar{a}_1) = 0$ by Formula (4.2-7), if $\bar{a}_1 \mapsto (0)$, $\bar{\mathfrak{A}}^{0,1}$ given by ① of (i) is obtained. If $r_0 > 1$, repeating the procedure in ① of (i), $\bar{\mathfrak{A}}^{0,r_0}$ is finally reached with the first arrow a_1^0 and $\delta(a_1^0) = 0$. If a_1^0 is \bar{a} -class and $a_1^0 \mapsto (1), (0 \ 1)$, ① of (iii) is obtained; if $a_1^0 = \bar{b}$ then $\bar{b} \mapsto W$, ② of (iii) is obtained, and $\kappa = 1$ in both cases.

Otherwise, if $a_1^0 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in case ② of (i), the induced pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$ is obtained, which satisfies (BRC) by Lemma 4.3.2–4.3.3.

Suppose $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ for some $i < \kappa - 1$ given in (i) has been obtained. Now we continue the reductions up to the induced pair $(\bar{\mathfrak{A}}^{i+1}, \bar{\mathfrak{B}}^{i+1})$. $(\bar{\mathfrak{A}}^i, \bar{\mathfrak{B}}^i)$ satisfies (BRC) by Lemma 4.3.2–4.3.3 inductively. Suppose the first arrow of $\bar{\mathfrak{B}}^{i,0}$ is $a_1^{i,0} = a_{\tau n^i q}$ or $b_{\tau n^i q}$ splitting from a non-effective arrow a_τ or b_τ with n^i being the index of the bottom row of $\bar{\Theta}^i$, and q being the column index inside the splitting block partitioned under $\bar{\mathcal{T}}$, then $\delta(a_1^{i,0}) = \underline{v}_{\tau n^i q}$, or $\bar{u}_{\tau n^i q}$ by Formulae (4.2-7)–(4.2-8), (4.1-8) and Remark 4.1.3 (i). Thus $a_1^{i,0} \mapsto \emptyset, \underline{v}_{\tau n^i q} = 0$ or $\bar{u}_{\tau n^i q} = 0$ by Remark 4.1.3 (ii). The regularizations are continue for the non-effective arrows inductively, and finally an effective one is sent to (0), then $\bar{\mathfrak{A}}^{i,1}$ is obtained by ① of (i). With a similar argument as above, $\bar{\mathfrak{A}}^{i,r_i}$ is reached, the first arrow of $\bar{\mathfrak{B}}^{i,r_i}$ has the differential $\delta(a_1^i) = 0$. Let $a_1^i \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the $(i + 1)$ -th pair is obtained.

If the procedure in (i) was continued without stop, the reduction sequence would have been infinite. Meanwhile, the \bar{b} -class loop at the down-right corner of the matrix block splitting from \bar{b} partitioned under \bar{T} has never been reached in (i). Therefore the procedure of (iii) must occur at some stage, say at the stage $\kappa - 1$.

If the first arrow $a_1^{\kappa-1}$ of $\bar{\mathfrak{B}}^{\kappa-1, r_{\kappa-1}}$ is an edge, then $a_1^{\kappa-1} \mapsto (1)$ or $(0\ 1)$ gives case ① of (iii). If $a_1^{\kappa-1}$ is a loop, then $a_1^{\kappa-1} \mapsto W$ by a loop reduction of Lemma 2.3.2 gives case ② of (iii), since $\bar{\mathfrak{A}}^{\kappa-1, r_{\kappa-1}}$ is not local from $\bar{\mathfrak{A}}^{\kappa-1, 0}$ being non-local by (BRC). In both cases, the induced pair $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ has the minimal index κ satisfying Condition (B). \square

Suppose $s(a_1^{i-1}) = X^{i-1}$ in case (i) ② of Theorem 4.3.4, the reduction on a_1^{i-1} gives $e_{X^{i-1}} \mapsto \begin{pmatrix} e_{Y^i} & \bar{w}^i \\ 0 & e_{X^i} \end{pmatrix}$ for $1 \leq i < \kappa$. Denote by \bar{W}_κ^i the split of \bar{w}^i in e_X^κ for $1 \leq i < \kappa$. Then \bar{W}_κ^i can be divided into $(\kappa - i)$ blocks $\bar{W}_{\kappa i}^i, \dots, \bar{W}_{\kappa, \kappa-1}^i$. Denote by n_i^κ the size of $e_{Y^i}^\kappa$, and by n_κ^κ that of $e_{X^\kappa}^\kappa$, which is 1 in case (iii) ① of Theorem 4.3.4, or is the same as that of W in (iii) ②. Thus $\bar{W}_{\kappa j}^i$ has the size $n_i^\kappa \times n_{j+1}^\kappa$. Write $n^\kappa = \sum_{i=1}^\kappa n_i^\kappa$, which is the size of e_X^κ . We have

$$e_X^\kappa = \begin{pmatrix} e_{Y^1}^\kappa & \bar{W}_{\kappa 1}^1 & \cdots & \cdots & \bar{W}_{\kappa, \kappa-1}^1 \\ & e_{Y^2}^\kappa & \cdots & \cdots & \bar{W}_{\kappa, \kappa-1}^2 \\ & & \cdots & \cdots & \\ & & & e_{Y^{\kappa-1}}^\kappa & \bar{W}_{\kappa, \kappa-1}^{\kappa-1} \\ & & & & e_{X^\kappa}^\kappa \end{pmatrix} \quad (4.3-2)$$

Corollary 4.3.5 The elements in \bar{W}_κ^i for $1 \leq i < \kappa$ are dotted arrows of $\bar{\mathfrak{B}}^\kappa$.

Proof The assertion is already implied in the proof of Theorem 4.3.4.

When we make an edge or a loop reduction of Lemma 2.3.2, the dotted arrows $\{F_i^{\prime*} \mid i = 1, \dots, l\}$, given in proof 2) of Proposition 2.1.5 are said to be *w-class arrows*, where the dotted arrows in \bar{W}_κ^i for $1 \leq i < \kappa$ of Formula (4.3-2) are specially said to be *\bar{w} -class*. Furthermore the elements splitting from w or \bar{w} -class arrows are still said to be in the same class.

4.4 Major pairs

We prove in this subsection that under some further assumption, a one sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1) is not homogeneous.

Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair, where $\bar{\mathfrak{B}}_X$ is given by Formula (4.2-1), and $\bar{\mathfrak{B}}$ has s \bar{a} -class arrows with $s \geq 1$. According to the coefficients of the first two Formulae of (4.2-8), s linear combinations of the \bar{v} -class arrows are define in $\bar{\mathfrak{B}}$:

$$\hat{v}_\tau = \sum_j (\bar{\varepsilon}_{\tau j} - \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\beta}_i \varepsilon_{i\tau j}) \bar{v}_j, \quad \tau = 1, \dots, s. \quad (4.4-1)$$

Fix any $1 \leq \tau \leq s$, making reductions according to (iii) ① of Theorem 4.3.4 for $\kappa = 1$, such that $a_1^0 = \bar{a}_\tau \mapsto (1)$, the induced pair $(\bar{\mathfrak{A}}^1, \bar{\mathfrak{B}}^1)$ is reached. Then we continue to do further reductions based on Formulae (4.2-7)–(4.2-8) inductively. For $\bar{a}_\eta \prec a_i, b_i \prec \bar{a}_{\eta+1}, \tau \leq \eta < s$ and $\bar{a}_s \prec a_i, b_i \prec \bar{b}$, by Remark 4.1.3 (ii):

$$a_i \mapsto \emptyset, \underline{v}_i + 1 \sum_j \varepsilon_{i\tau j} \underline{v}_j = 0, \quad b_i \mapsto \emptyset, \bar{u}_i + 1 \sum_j \varepsilon_{i\tau j} \bar{v}_j = 0. \quad (4.4-2)$$

On the other hand, $\bar{a}_\eta \mapsto \emptyset$ or (0) for $\tau < \eta \leq s$ corresponding to $\delta(\bar{a}_{\tau+\eta}) \neq 0$ or $= 0$. the dotted element \bar{u}_i is replaced by the linear composition of \bar{v} -class arrows inductively by Remark 4.1.3 (iii), the second formula of (4.2-8) shows the formula below in some induced pair:

$$\begin{aligned} \delta(\bar{b}) &= \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\beta}_i \bar{u}_i + 1(\sum_j \bar{\varepsilon}_{\tau j} \bar{v}_j) \\ &= \sum_j (\bar{\varepsilon}_{\tau j} - \sum_{\bar{a}_\tau \prec b_i \prec \bar{b}} \bar{\beta}_i \varepsilon_{i\tau j}) \bar{v}_j = \hat{v}_\tau. \end{aligned} \quad (4.4-3)$$

Lemma 4.4.1 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathcal{T}}$ being trivial, $s \geq 1$, and $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1). If there exists some $1 \leq \tau \leq s$, with $\hat{v}_\tau = 0$ in Formula (4.4-1), then $\bar{\mathfrak{B}}$ is wild and non-homogeneous.

Proof If $\bar{a}_\tau : X \rightarrow Y$, it may be assumed that $\mathcal{T} = \{X, Y\}$.

1) Since $\bar{\mathfrak{B}}_X$ is minimal with $R_X = k[x, \phi(x)^{-1}]$, there is an almost split conflation $(e'_\lambda) : S'_\lambda \rightarrow E'_\lambda \rightarrow S'_\lambda$ for any $\lambda \in \mathcal{L}' = k \setminus \{\text{roots of } \phi(x)\}$ in $R(\bar{\mathfrak{B}}_X)$. Let $\vartheta : R(\bar{\mathfrak{B}}_X) \rightarrow R(\bar{\mathfrak{B}})$ be the induced functor. If $\bar{\mathfrak{B}}$ is homogeneous, then there is a co-finite subset $\mathcal{L} \subseteq \mathcal{L}'$, and a set of almost split conflations $\{(e_\lambda) = \vartheta(e'_\lambda) : S_\lambda \rightarrow E_\lambda \rightarrow S_\lambda \mid \lambda \in \mathcal{L}\}$ by Corollary 3.2.4.

2) According to Formula (4.4-1)-(4.4-3), an induced pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ is obtained with $\delta(\bar{b}) = 0$. Thus it is possible to construct an object $L \in R(\bar{\mathfrak{B}})$ with $L_X = k, L_Y = k, L(\bar{a}_\tau) = (1), L(\bar{b}) = (\lambda)$ and $L(b_i) = 0, i > n_0, L(c_i) = 0, i = 1, \dots, t$. The same argument given in 2)-3) of the proof of Lemma 3.4.1 shows that $\bar{\mathfrak{B}}$ is non-homogeneous. And $\bar{\mathfrak{B}}$ is wild by [CB1, Theorem A]. \square

Theorem 4.4.2 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair with $\bar{\mathcal{T}}$ being trivial, $s > 1$, and $\bar{\mathfrak{B}}_X$ given by Formula (4.2-1). If the elements $\{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_s\}$ defined in Formula (4.4-1) are linearly dependent, then $\bar{\mathfrak{B}}$ is wild and non-homogeneous.

Proof Without loss of generality, it may be assumed $\bar{\mathcal{T}} = \{X, Y\}$. Suppose there is a minimal linearly dependent subset $\{\hat{v}_{\tau_1}, \hat{v}_{\tau_2}, \dots, \hat{v}_{\tau_l}\}$ with l vectors. Since the case of $l = 1$ has been treated in Lemma 4.4.1, it is assumed here that $l > 1$. Suppose $\tau_1 < \tau_2 < \dots < \tau_l$ and

$$\hat{v}_{\tau_1} = \beta_2 \hat{v}_{\tau_2} + \dots + \beta_l \hat{v}_{\tau_l}, \quad \beta_2, \dots, \beta_l \in k^*. \quad (4.4-5)$$

1) Making reductions according to Theorem 4.3.4 (i) and (iii) ① for $\kappa = l$, such that $a_1^{p-1} \mapsto \binom{1}{0}$ for $1 \leq p < l$, and $a_1^l \mapsto (1)$, an induced pair $(\bar{\mathfrak{A}}^l, \bar{\mathfrak{B}}^l)$ is obtained. The sum $F^l + \bar{\Theta}^l$ looks like (with only \bar{a}, \bar{b}, c -class arrows):

0	0	\dots	$\bar{a}_{\tau_2,1}$	\dots	$\bar{a}_{\tau_3,1}$	\dots	$\bar{a}_{\tau_l,1}$	$\bar{a}_{\tau_l+1,1} \dots \bar{a}_{s1}$	$\bar{b}_{11} \ \bar{b}_{12} \ \dots \ \bar{b}_{1l}$	$c_{11} \ \dots \ c_{t1}$	(4.4-6)
			1	\dots	$\bar{a}_{\tau_3,2}$	\dots	$\bar{a}_{\tau_l,2}$	$\bar{a}_{\tau_l+1,2} \dots \bar{a}_{s2}$	$\bar{b}_{21} \ \bar{b}_{22} \ \dots \ \bar{b}_{2l}$	$c_{12} \ \dots \ c_{t2}$	
		0		1	\dots	$\bar{a}_{\tau_l,3}$	$\bar{a}_{\tau_l+1,3} \dots \bar{a}_{s3}$	$\bar{b}_{31} \ \bar{b}_{32} \ \dots \ \bar{b}_{3l}$	$c_{13} \ \dots \ c_{t3}$		
				\dots			\dots				
			0		1	$\bar{a}_{\tau_l+1,l} \dots \bar{a}_{sl}$	$\bar{b}_{l1} \ \bar{b}_{l2} \ \dots \ \bar{b}_{ll}$	$c_{1l} \ \dots \ c_{tl}$			

Since $\bar{\mathfrak{A}}$ has two vertices, the dimension of $\vartheta^{0l}(F^l(k))$ in $R(\bar{\mathfrak{A}})$ is $l + 1$, and the number of links of F^l is l , the pair $(\bar{\mathfrak{A}}^l, \bar{\mathfrak{B}}^l)$ is local by the assertion below Formula (2.3-7).

2) We make further reductions from $\bar{\mathfrak{B}}^l$ inductively for the \bar{p} -th row ordered by $\bar{p} = l, l - 1, \dots, 2$ in the reduced formal product $\bar{\Theta}^l$. For $\bar{p} = l$, similar to Formulae (4.4-2)-(4.4-3): $a_{il} \mapsto \emptyset, i \in \Lambda$; note that $\bar{v}_j : Y \rightarrow X$, the matrix splitting from \bar{v}_j in $\bar{\Psi}_{\underline{n}^l}$ is $(\bar{v}_{j1}, \dots, \bar{v}_{jl})$ of size $1 \times l$, $b_{ilq} \mapsto \emptyset, \bar{u}_{ilq} + 1 \sum_j \varepsilon_{i\tau_l j} \bar{v}_{jq} = 0, i < n_0; \bar{a}_{\eta l} \mapsto (0)$ or $\emptyset, \tau_l < \eta \leq s$;

$$\begin{aligned} \delta^0(\bar{b}_{lq}) &= \sum_{\bar{a}_{\tau_l} \prec b_i \prec \bar{b}} \bar{\beta}_i \bar{u}_{ilq} + 1(\sum_j \bar{\varepsilon}_{\tau_l j} \bar{v}_{jq}) \\ &= \sum_j (\bar{\varepsilon}_{\tau_l j} - \sum_{\bar{a}_{\tau_l} \prec b_i \prec \bar{b}} \bar{\beta}_i \varepsilon_{i\tau_l j}) \bar{v}_{jq} = \hat{v}_{\tau_l q}, \end{aligned}$$

thus $\bar{b}_{lq} \mapsto \emptyset, \hat{v}_{\tau_l q} = 0$ for $q = 1, \dots, l$ inductively. Next, $b_{ilq} \mapsto \emptyset$ for $i > n_0, 1 \leq q \leq l$, by Remark 4.1.3 (ii); and $c_{il} \mapsto (0)$ or \emptyset . The dotted arrows $\underline{v}_{ip}, \underline{u}_{ipq}$ for all i and $p < l, 1 \leq q \leq l$, \hat{v}_{iq} for $i < l$ and $1 \leq q \leq l$ are preserved by 4.1.3 (iv). The induced boc $\bar{\mathfrak{B}}^{l+1}$ follows.

3) Suppose an induced boc $\bar{\mathfrak{B}}^{2l-\bar{p}}$ is reached for some $\bar{p} < l$. Denote the entries of $F^{2l-\bar{p}}$, which are not the entries of $G_{2l-\bar{p}}^j$ for $j = 1, \dots, l$, by \bullet^0 coming from \bullet , one of the a, b, c -class solid arrows. Then $\bar{\mathfrak{B}}^{2l-\bar{p}}$ satisfies the following two conditions.

- ① for any $p > \bar{p}$, $a_{ip}^0 = \emptyset, i \in \Lambda; \bar{a}_{ip}^0 = \emptyset$ or $(0); b_{ip}^0 = \emptyset, i \neq n_0; \bar{b}_{pq} = \emptyset; c_{ip} = \emptyset$ or (0) .
- ② The dotted arrows $\underline{v}_{ip}, \underline{u}_{ipq}$ for all i, q and $p \leq \bar{p}$; and \hat{v}_{iq} for $i \leq \bar{p}$ and $1 \leq q \leq l$ are preserved.

In $A_{\zeta}^j, B_{\zeta}^j, C_{\zeta}^j$, a solid arrow is denoted by \bullet_{ipq}^j splitting from a_i, b_i, c_i respectively, an entry of F^{ζ} by $\zeta_{i,pq}^{j,0}$, where (p, q) is the index in the $n^{\zeta} \times n_{t(a_i)}^{\zeta}, n^{\zeta} \times n^{\zeta}$ or $n^{\zeta} \times n_{t(c_i)}^{\zeta}$ -block matrices. For the sake of convenience, $\Phi_{\underline{m}_{\zeta}}^m$ of Formula (4.1-7) is also partitioned by the lines $m_j^{\zeta}, l_j^{\zeta}, r_j^{\zeta}$ in the same way as in $F^{\zeta} + \Theta^{\zeta}$.

Remark From now on, the pseudo formal equation (4.1-7) at the ς -th step for presenting the differential of the first arrow is considered. After a loop or an edge reduction, some w -class dotted arrows may be added into the induced boc, but the linear relations among the splits of the dotted elements in the induced pair can be obtained completely by Remark 4.1.3 (i). Therefore the new relation of $\bar{u}, \bar{v}, \underline{u}, \underline{v}, \bar{w}, w$ -class elements during the regularization from $\bar{\mathfrak{B}}^\varsigma$ to $\bar{\mathfrak{B}}^{\varsigma+1}$ will be concentrated on.

Throughout the subsection, suppose the first arrow $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\iota$ of $\bar{\mathfrak{B}}^\varsigma$ belongs to $A_\varsigma^\iota \cup B_\varsigma^\iota \cup C_\varsigma^\iota$ in Picture (4.5-1). Denote by \bar{u}_{ipq}^j or \underline{v}_{ipq}^j the dotted element in $\bar{\Psi}_{\underline{n}^\varsigma}^m$, which corresponds to the entry $b_{ipq}^{j,0}$ or $a_{ipq}^{j,0}$ in F^ς , thus $j \geq \iota$, $p > \bar{p}$, or $p = \bar{p}$ but $q < \bar{q}$. On the other hand, denote by $\bar{u}_{i'p'q'}^{j'}$ or $\underline{v}_{i'p'q'}^{j'}$ the dotted element corresponding to the solid arrow $b_{i'p'q'}^{j'}$ or $a_{i'p'q'}^{j'}$ in Θ^ς , thus $j' \leq \iota$, and $p' < \bar{p}$ or $p' = \bar{p}, q' \geq \bar{q}$. All of them are coming from $(\mathcal{D}, \mathcal{U})$ in Lemma 4.3.3,

In the following Lemmas, the index n_0 is defined in Formula (4.2-1), and the number h and the set Λ are defined in Formula (4.2-5).

Lemma 4.5.1 Let $(\mathfrak{A}^\varsigma, \mathfrak{B}^\varsigma)$ be a pair induced from $(\mathfrak{A}^\kappa, \mathfrak{B}^\kappa)$ given by Theorem 4.3.4 (iii) ②. Suppose the first arrow of \mathfrak{B}^ς , $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\kappa \in B_\varsigma^\kappa \cup C_\varsigma^\kappa$, (see the second thick line below in Picture (4.5-1) for example). Assume that

- (i) all $b_{ipq}^{\kappa,0} = \emptyset, i > n_0$, and the corresponding dotted element \bar{u}_{ipq}^κ is replaced by a linear combination of some \bar{v} -class arrows in $\bar{\mathfrak{B}}^\varsigma$; while the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved;
- (ii) if $c_{ipq}^{\kappa,0} = \emptyset$, there is a linear relation among some elements $\underline{v}_{i_1p_1q}^\kappa, h < i_1 \leq h + i, p_1 \geq p$ and some \underline{u}, w -class arrows in $\bar{\mathfrak{B}}^\varsigma$; while all the dotted arrows $\underline{v}_{i'p'q'}^{j'}$ are preserved.

Then after a regularization, the induced pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies (i)-(ii). In particular all the dotted arrows $\underline{v}_{i'p'q'}^{j'}$ except $\underline{v}_{\tau\bar{p}\bar{q}}^\kappa$ in case (ii); $\bar{u}_{i'p'q'}^{j'}$, except $\bar{u}_{\tau\bar{p}\bar{q}}^\kappa$ in case (i); and all the \bar{w}, \bar{v} -class arrows are preserved.

Proof The assumption (i)-(ii) are valid for $\varsigma = \kappa$ by Theorem 4.3.4 and Corollary 4.3.5.

- (i) If $a_1^\varsigma = b_{\tau\bar{p}\bar{q}}^\kappa, \tau > n_0$, then according to the third formula of (4.2-8) and Formula (4.1-8),

$$\delta(b_{\tau\bar{p}\bar{q}}^\kappa) = \bar{u}_{\tau\bar{p}\bar{q}}^\kappa + \sum_{n_0 < i < \tau} (\beta_{\tau i}^0 \bar{u}_{i\bar{p}\bar{q}}^\kappa + \sum_q \beta_{\tau i}^1 \bar{b}_{\bar{p}q}^{\kappa,0} \bar{u}_{iq\bar{q}}^\kappa) + \sum_{c_i < b_{\tau,q}} c_{i\bar{p}q}^{\kappa,0} (\sum_l \varepsilon'_{\tau il} \bar{v}_{lq\bar{q}}).$$

Since W_ς^κ is upper triangular, the index $\bar{p} \leq q$ in $\bar{b}_{\bar{p}q}^{\kappa,0}$. By assumption (i), $\bar{u}_{\tau\bar{p}\bar{q}}^\kappa$ is a dotted arrow, thus $b_{\tau\bar{p}\bar{q}}^\kappa \mapsto \emptyset$, $\bar{u}_{\tau\bar{p}\bar{q}}^\kappa$ is replaced by a linear combination of some \bar{v} -class arrows by 4.1.3 (ii)-(iii), since $\bar{u}_{i\bar{p}\bar{q}}^\kappa, \bar{u}_{iq\bar{q}}^\kappa$ are already replaced by those arrows still by assumption (i).

- (ii) If $a_1^\varsigma = c_{\tau\bar{p}\bar{q}}^\kappa$, then according to Formula (4.2-9) and (4.1-8),

$$\delta(c_{\tau\bar{p}\bar{q}}^\kappa) = \sum_{h < i \leq h + \tau} (\gamma_{\tau i}^0 \underline{v}_{i\bar{p}\bar{q}}^\kappa + \sum_q \gamma_{\tau i}^1 \bar{b}_{\bar{p}q}^{\kappa,0} \underline{v}_{iq\bar{q}}^\kappa) + \sum_{i,q} c_{i\bar{p}q}^{\kappa,0} (\sum_l \xi_{\tau il} \underline{u}_{lq\bar{q}}) + \sum_{q < \bar{q}} c_{\tau\bar{p}q}^{\kappa,0} w_{q\bar{q}} - \sum_{p > \bar{p}} w_{\bar{p}p} c_{\tau\bar{p}q}^0.$$

In the case of $\delta(c_{\tau\bar{p}\bar{q}}^\kappa) \neq 0$, $c_{\tau\bar{p}\bar{q}}^\kappa \mapsto \emptyset$, and a linear relation among elements $\underline{v}_{i\bar{p}\bar{q}}, \underline{v}_{iq\bar{q}}, h < i \leq h + \tau, q \geq \bar{p}$, and some \underline{u}, w -class elements is added, which is given by the right-hand side of the above formula being equal to 0.

The required \underline{v}, \bar{w} -class and all the \bar{v}, \bar{w} -class dotted arrows are preserved, the pair $(\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1})$ still satisfies assumption (i)-(ii). \square

If γ exists in Theorem 4.3.4 (ii), suppose $\gamma < j \leq \kappa$ in case (iii) ① of 4.3.4, or $\gamma < j < \kappa$, in case (iii) ②, see what between the two thick lines of Picture (4.5-1) for example.

A \bar{w} -class elements of e_X^ς is described by two indices \bar{w}_{pq} , where p is the row index and q the column index of e_X^ς . Suppose the identity matrix I_ς^j intersects the p -th row of F^ς at the q_p^j -th column with $\bar{b}_{pq_p^j}^{j,0} = (1)$. Denote by $\bar{w}_{q_p^j q}$ for any possible q the dotted element with the row index q_p^j in e_X^ς . Let $\gamma < \iota \leq \kappa$ (or $< \kappa$) be given above. If $p > \bar{p}$, then $\bar{w}_{q_p^j q}$ is sitting below $\bar{w}_{q_p^\kappa q}$. Note that \bar{W}_ς^j , the j -block in e_X^ς , defined by Formula (4.3-2) is also sitting in the (n_0, n_0) -th block of

$\bar{\Psi}_{\underline{n}^\varsigma}^r$ partitioned under $\bar{\mathcal{T}}$. Clearly, the row indices of \bar{W}_ζ^j coincide with the column indices of I_ζ^j , see $(I^3, \bar{W}_3), (I^4, \bar{W}_4)$ in Picture (4.5-2) at the end of the subsection.

Lemma 4.5.2 Let $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ be an induced pair of $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ with γ existing in Theorem 4.3.4

- (ii). Suppose the first arrow of $\bar{\mathfrak{B}}^\varsigma$, $a_1^\varsigma = \bullet_{\tau\bar{p}\bar{q}}^\iota \in B_\zeta^\iota \cup C_\zeta^\iota$ with $\gamma < \iota \leq \kappa$ (or $< \kappa$). Assume that
- (i) all $\bar{b}_{pq}^{j,0} = \emptyset$, the corresponding dotted element $\bar{w}_{q_p^j}$ is replaced by a linear combination of some \bar{w}_{pq} for $p > q_p^j$, and some w -class elements in $\bar{\mathfrak{B}}^\varsigma$; while the dotted arrows $\bar{w}_{p'q'}$ for $p' \leq q_p^j$ are preserved;
 - (ii) all $\bar{b}_{ipq}^{j,0} = \emptyset, i > n_0$, the corresponding \bar{u}_{ipq}^j is replaced by a linear combination of an element $\bar{u}_{i_1p_1q}^{j_1}$, with $n_0 < i_1 < i, j \leq j_1 < \gamma, p_1 < p$, and some \bar{v} -class elements in $\bar{\mathfrak{B}}^\varsigma$; while the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved;
 - (iii) if $\bar{c}_{ipq}^{j,0} = \emptyset$, there is a linear relation among some elements $\bar{v}_{i_1p_1q}^{j_1}, h < i_1 \leq h + i, j \leq j_1 < \gamma, p_1 \leq p$, and some \underline{u}, w -class elements in $\bar{\mathfrak{B}}^\varsigma$; while the dotted arrow $\bar{v}_{i'p'q'}^{j'}, i' \in \Lambda$, are preserved.

Then after a regularization, the induced pair $(\bar{\mathfrak{A}}^{\varsigma+1}, \bar{\mathfrak{B}}^{\varsigma+1})$ still satisfies (i)-(iii). In particular all the dotted arrows $\bar{v}_{i'p'q'}^{j'}, i' \in \Lambda$ in case (iii); $\bar{u}_{i'p'q'}^{j'}$ except $\bar{u}_{\tau\bar{p}\bar{q}}^j$ in case (ii); $\bar{w}_{p'q'}, p' < q_p^j$, in case (i); and all the \bar{v} -class dotted arrows are preserved.

Proof The assumption (i)-(iii) are valid in the following two cases. First, the box of a_1^ς has the bottom and right boundaries $(m_\zeta^\kappa, r_\zeta^\kappa)$ in case (iii) ① of Theorem 4.3.4. Second, a_1^ς has those boundaries $(m_\zeta^{\kappa-1}, r_\zeta^{\kappa-1})$ in case (iii) ② of 4.3.4 according to Lemma 4.5.1.

- (i) If $a_1^\varsigma = \bar{b}_{\bar{p}\bar{q}}^\iota$, then $\bar{b}_{\bar{p}\bar{q}}^{\iota,0} = (1)$, $\bar{b}_{\bar{p}\bar{q}}^\iota = \emptyset, q_p^\iota < q < \bar{q}$, by assumption (i), Formula (4.2-8) shows

$$\delta(\bar{b}_{\bar{p}\bar{q}}^\iota) = 1\bar{w}_{q_p^\iota\bar{q}} - \sum_{j \geq \iota, q > \bar{p}} w_{\bar{p}q} \bar{b}_{q\bar{q}}^{j,0},$$

where $w_{\bar{p}p}$ is \bar{w} or w -class. Since $\bar{w}_{q_p^\iota\bar{q}}^\iota$ is a dotted arrow still by (i), $\bar{b}_{\bar{p}\bar{q}}^\iota \mapsto \emptyset$ and $\bar{w}_{q_p^\iota\bar{q}}^\iota$ is replaced by a linear combination of some \bar{w} -class elements below the q_p^ι -th row and some w -class elements in the pair $(\bar{\mathfrak{A}}^{\varsigma+1}, \bar{\mathfrak{B}}^{\varsigma+1})$.

- (ii) If $a_1^\varsigma = b_{\tau\bar{p}\bar{q}}^\iota, \tau > n_0$, then $\bar{b}_{\bar{p}\bar{q}}^{\iota,0} = (1)$, $\bar{b}_{\bar{p}\bar{q}}^\iota = \emptyset, \forall q_p^\iota < q < \bar{q}$, Formula (4.2-8) gives

$$\delta(b_{\tau\bar{p}\bar{q}}^\iota) = \bar{u}_{\tau\bar{p}\bar{q}}^\iota + \sum_{n_0 < i < \tau} (\beta_{\tau i}^0 \bar{u}_{i\bar{p}\bar{q}}^\iota + \beta_{\tau i}^1 \bar{b}_{\bar{p}q_p^\iota}^{\iota,0} \bar{u}_{iq_p^\iota\bar{q}}^{\iota_1}) + \sum_{c_i < b_{\tau,q}} c_{i\bar{p}\bar{q}}^0 (\sum_l \varepsilon'_{\tau il} \bar{v}_{lq\bar{q}}),$$

where $\iota \leq \iota_1 < \gamma, q_p^\iota < \bar{p}$. Since $\bar{u}_{\tau\bar{p}\bar{q}}^\iota$ is a dotted arrow by assumption (ii), $b_{\tau\bar{p}\bar{q}}^\iota \mapsto \emptyset$, and $\bar{u}_{\tau\bar{p}\bar{q}}^\iota$ is replaced by a \bar{u} -class element and some \bar{v} -class elements by Remark 4.1.3 (ii)-(iii).

- (iii) If $a_1^\varsigma = c_{\tau\bar{p}\bar{q}}^\iota$, then according to Formula (4.2-9):

$$\begin{aligned} \delta(c_{\tau\bar{p}\bar{q}}^\iota) &= \sum_{h < i \leq h+\tau, \iota_1 < \iota} (\gamma_{\tau i}^0 \bar{v}_{i\bar{p}\bar{q}}^\iota + \gamma_{\tau i}^1 \bar{b}_{\bar{p}q_p^\iota}^{\iota,0} \bar{v}_{iq_p^\iota\bar{q}}^{\iota_1}) \\ &\quad + \sum_{i,q} c_{i\bar{p}\bar{q}}^{\iota,0} (\sum_l \xi_{\tau il} \bar{u}_{lq\bar{q}}) + \sum_{q < \bar{q}} \bar{c}_{\tau\bar{p}q}^{\iota,0} w_{q\bar{q}} - \sum_{p > \bar{p}} w_{\bar{p}p} \bar{c}_{\tau\bar{p}q}^{j,0}, \end{aligned}$$

where $\iota \leq \iota_1 < \gamma, q_p^\iota < \bar{p}$. If $\bar{c}_{\tau\bar{p}\bar{q}}^\iota \mapsto \emptyset$, a linear relation among some \underline{u}, w -class elements and some \underline{v} -class elements with the subscripts being bigger than h is added.

The required $\underline{v}, \bar{u}, \bar{w}$ -class and all the \bar{v} -class dotted arrows are preserved, and the pair $(\bar{\mathfrak{A}}^{\varsigma+1}, \bar{\mathfrak{B}}^{\varsigma+1})$ still satisfies assumption (i)-(iii). \square

Suppose $j \leq \gamma$ if γ exists, otherwise $j \leq \kappa$ in case (iii) ① of Theorem 4.3.4, or $j < \kappa$ in (iii) ②, see the first thick line above in Picture (4.5-1) for example.

Assume I_ζ^j intersects the p -th row of F^ς at the q_p^j -th column in the $n_X^\varsigma \times n_{t(a_{ij})}^\varsigma$ -block obtained from \bar{a}_{ij} partitioned under $\bar{\mathcal{T}}$ with $\bar{a}_{ijpq_p^j}^{j,0} = (1)$. Denote by $\hat{v}_{ijq_p^j}$ the (q_p^j, q) -element in the block of size $n_{t(a_{ij})}^\varsigma \times n_X^\varsigma$ splitting from \hat{v}_{ij} . Denote by \hat{V}_ζ^j the block in the n_0 -th block-column of $\bar{\Psi}_{\underline{n}^\varsigma}^r$

partitioned under $\bar{\mathcal{T}}$, such that the row indices of \hat{V}_ζ^j coincide with the column indices of I_ζ^j , see $(I^1, \hat{V}_1), (I^2, \hat{V}_2)$ in Picture (4.5-2) below.

Lemma 4.5.3 Let $(\bar{\mathfrak{A}}^\zeta, \bar{\mathfrak{B}}^\zeta)$ be an induced pair of $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$. Suppose the first arrow of $\bar{\mathfrak{B}}^\zeta$, $a_1^\zeta = \bullet_{\tau\bar{p}\bar{q}}^\iota \in A_\zeta^\iota \cup B_\zeta^\iota \cup C_\zeta^\iota$, where $\iota \leq \gamma$ if γ exists, otherwise $\iota \leq \kappa$ in case (iii) ① of Theorem 4.3.4, or $\iota < \kappa$ in (iii) ②. Assume that

- (i) all $a_{ipq}^{j,0} = \emptyset, i \in \Lambda$, the corresponding \underline{v}_{ipq}^j is replaced by a linear combination of some \underline{u} -class elements in $\bar{\mathfrak{B}}^\zeta$; while all the dotted arrows $\underline{v}_{i'p'q'}^{j'}$ are preserved;
 - (ii) if $\bar{a}_{ipq}^{j,0} = \emptyset$, there is a linear relation among $\underline{u}, \bar{w}, w$ -class elements in $\bar{\mathfrak{B}}^\zeta$; while all the dotted arrows $\underline{v}_{i'p'q'}^{j'}, i' \in \Lambda$, are preserved;
 - (iii) all $b_{ipq}^{j,0} = \emptyset, i < n_0$, the corresponding \bar{u}_{ipq}^j is replaced by a linear combination of some \bar{v} -class elements in $\bar{\mathfrak{B}}^\zeta$; while all the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved;
 - (iv) all $\bar{b}_{pq}^{j,0} = \emptyset, \hat{v}_{ijq_{pq}^j}$ corresponding to $\bar{a}_{ijpq}^{j,0}$ is replaced by a linear combination of some \bar{v} -class elements below and some \bar{w}, w -class in $\bar{\mathfrak{B}}^\zeta$; while the dotted arrows $\hat{v}_{i'p'q'}, p' \leq q_{\bar{p}}^\iota$, are preserved;
 - (v) all $b_{ipq}^{j,0} = \emptyset, i > n_0$, the corresponding element \bar{u}_{ipq}^j is replaced by a linear combination of some \bar{v} -class elements in $\bar{\mathfrak{B}}^\zeta$; while all the dotted arrows $\bar{u}_{i'p'q'}^{j'}$ are preserved;
 - (vi) if $c_{ipq}^{j,0} = \emptyset$, there is a linear relation among some elements $\underline{v}_{i_1p_1q}^{j_1}, h < i_1 < h + \tau, p_1 = p$, and some $\underline{u}, w, \bar{w}$ -class elements in $\bar{\mathfrak{B}}^\zeta$; while all the dotted arrows $\underline{v}_{i'p'q'}^{j'}, i' \in \Lambda$, are preserved.
- Then after a regularization, the induced pair $(\bar{\mathfrak{A}}^{\zeta+1}, \bar{\mathfrak{B}}^{\zeta+1})$ still satisfies (i)-(vi). In particular, all the dotted arrows $\underline{v}_{i'p'q'}^{j'}, i' \in \Lambda$, except $\underline{v}_{\tau\bar{p}\bar{q}}^j$ in cases (i), (ii), (vi); $\bar{u}_{i'p'q'}^{j'}$ except $\bar{u}_{\tau\bar{p}\bar{q}}^j$ in case (iii) or (v) and $\hat{v}_{p'q'}^{j'}$ for $p' < q_{\bar{p}}^\iota$ in case (iv), are preserved.

Proof We claim first, that if γ exists, then $\bar{u}_{i_1p_1q}^{j_1} \succ \bar{u}_{ipq}^j$ given in case (ii) of Lemma 4.5.2 can be replaced inductively by some \bar{v} -class arrows, when the reductions inside the $(\gamma + 1)$ -th block in Picture (4.5-1) are finished. Thus the assumption (i)-(vi) are valid, if a_1^ζ has the bottom and right boundaries $(m_\zeta^\gamma, r_\zeta^\gamma)$, when γ exists, according to Lemma 4.5.2; otherwise a_1^ζ has those $(m_\zeta^\kappa, r_\zeta^\kappa)$ in case (iii) ① of Theorem 4.3.4; or $(m_\zeta^{\kappa-1}, r_\zeta^{\kappa-1})$ in (iii) ② by Lemma 4.5.1.

(i) If $a_1^\zeta = a_{\tau\bar{p}\bar{q}}^\iota, \tau \in \Lambda$, $\underline{v}_{\tau\bar{p}\bar{q}}^\iota$ is a dotted arrow by assumption (i). Formula (4.2-7) tells

$$\delta(a_{\tau\bar{p}\bar{q}}^\iota) = \underline{v}_{\tau\bar{p}\bar{q}}^\iota + \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \epsilon_{\tau il} \underline{u}_{lq\bar{q}}) \implies a_{\tau\bar{p}\bar{q}}^\iota \mapsto \emptyset, \quad \underline{v}_{\tau\bar{p}\bar{q}}^\iota = - \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \epsilon_{\tau il} \underline{u}_{lq\bar{q}}).$$

(ii) If $a_1^\zeta = \bar{a}_{\tau pq}$ is effective, then by substituting $\underline{v}_{i'p\bar{q}}^\iota$ given by the formula above,

$$\begin{aligned} \delta(\bar{a}_{\tau pq}^\iota) &= \sum_{\bar{a}_{\tau\iota} \prec a_{i'} \prec \bar{a}_\tau} \bar{\alpha}_{\tau i'} \underline{v}_{i'p\bar{q}}^\iota + \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \bar{\epsilon}_{\tau il} \underline{u}_{lq\bar{q}}) + \sum_{q < \bar{q}} \bar{a}_{\tau pq}^{\iota,0} w_{q\bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{a}_{\tau pq}^{j,0} \\ &= \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l (\bar{\epsilon}_{\tau il} - \sum_{\bar{a}_{\tau\iota} \prec a_{i'} \prec \bar{a}_\tau} \bar{\alpha}_{\tau i'} \epsilon_{i' il})) \underline{u}_{lq\bar{q}} + \sum_{q < \bar{q}} \bar{a}_{\tau pq}^{\iota,0} w_{q\bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{a}_{\tau pq}^{j,0}. \end{aligned}$$

If $\bar{a}_{\tau pq}^\iota \mapsto \emptyset$, then a linear relation among some \underline{u}, w and \bar{w} -class elements is added.

(iii) If $a_1^\zeta = b_{\tau\bar{p}\bar{q}}^\iota, \tau < n_0$, $\bar{u}_{\tau\bar{p}\bar{q}}^\iota$ is a dotted arrow by assumption (iii):

$$\delta(b_{\tau\bar{p}\bar{q}}^\iota) = \bar{u}_{\tau\bar{p}\bar{q}}^\iota + \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \epsilon_{\tau il} \bar{v}_{lq\bar{q}}) \implies b_{\tau\bar{p}\bar{q}}^\iota \mapsto \emptyset, \quad \bar{u}_{\tau\bar{p}\bar{q}}^\iota = - \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \epsilon_{\tau il} \bar{v}_{lq\bar{q}}).$$

(iv) If $a_1^\zeta = \bar{b}_{p\bar{q}}^\iota$ is effective, by substituting $\bar{u}_{i'p\bar{q}}^\iota$ given in (iii) and Formula (4.4-1),

$$\begin{aligned} \delta(\bar{b}_{p\bar{q}}^\iota) &= \sum_{i' < n_0} \bar{\beta}_{i'} \bar{u}_{i'p\bar{q}}^\iota + \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l \bar{\epsilon}_{il} \bar{v}_{lq\bar{q}}) + \sum_{q < \bar{q}} \bar{b}_{p\bar{q}}^{\iota,0} w_{q\bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{p\bar{q}}^{j,0} \\ &= \sum_{i,q} \bar{a}_{ipq}^{\iota,0} (\sum_l (\bar{\epsilon}_{il} - \sum_{\bar{a}_{\tau\iota} \prec \bar{a}_{i'} \prec b_i \prec \bar{b}} \bar{\beta}_{i'} \epsilon_{i' il})) \bar{v}_{lq\bar{q}} + \sum_{q < \bar{q}} \bar{b}_{p\bar{q}}^{\iota,0} w_{q\bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{p\bar{q}}^{j,0} \\ &= \hat{v}_{\tau\iota} q_{\bar{p}\bar{q}}^\iota + \sum_{\bar{a}_{ipq}^{\iota,0} \succ \bar{a}_{\tau\iota} q_{\bar{p}\bar{q}}^\iota} \bar{a}_{ipq}^{\iota,0} \hat{v}_{iq\bar{q}} + \sum_{q < \bar{q}} \bar{b}_{p\bar{q}}^{\iota,0} w_{q\bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p}p} \bar{b}_{p\bar{q}}^{j,0}. \end{aligned}$$

Since $\bar{a}_{\tau^\iota \bar{p} q \bar{p}}^{\iota, 0} = 1$, and $\hat{v}_{\tau^\iota q \bar{p} \bar{q}}$ is a dotted arrow by assumption (iv), $\bar{b}_{\bar{p} \bar{q}}^\iota \mapsto \emptyset$, and $\hat{v}_{\tau^\iota q \bar{p} \bar{q}}$ is replaced by some \hat{v} -class elements below the $q \bar{p}$ -th row, and some \bar{w}, w -class elements.

(v) If $a_1^\zeta = b_{\tau \bar{p} \bar{q}}^\zeta$, $\tau > n_0$, since $\bar{b}_{\bar{p} q}^0 = \emptyset$ for all possible q by (iv) above, Formula (4.2-8) shows

$$\delta(b_{\tau \bar{p} \bar{q}}^\zeta) = \bar{u}_{\tau \bar{p} \bar{q}}^\zeta + \sum_{i \neq n_0, i < \tau} \beta_{\tau i}^0 \bar{u}_{i \bar{p} \bar{q}}^\zeta + \sum_{i, q} \bar{a}_{i \bar{p} q}^{\iota, 0} (\sum_l \varepsilon_{\tau i l} \bar{v}_{l q \bar{q}}) + \sum_{c_i < b_{\tau; q}} \bar{c}_{i \bar{p} q}^{\iota, 0} (\sum_l \varepsilon'_{\tau i l} \bar{v}_{l q \bar{q}}).$$

Since $\bar{u}_{\tau \bar{p} \bar{q}}^\zeta$ is a dotted arrow by assumption (v), $b_{\tau \bar{p} \bar{q}}^\zeta \mapsto \emptyset$, $\bar{u}_{\tau \bar{p} \bar{q}}^\zeta$ is replaced by some \bar{v} -class elements according to the replacement given in (iii) and assumption (v).

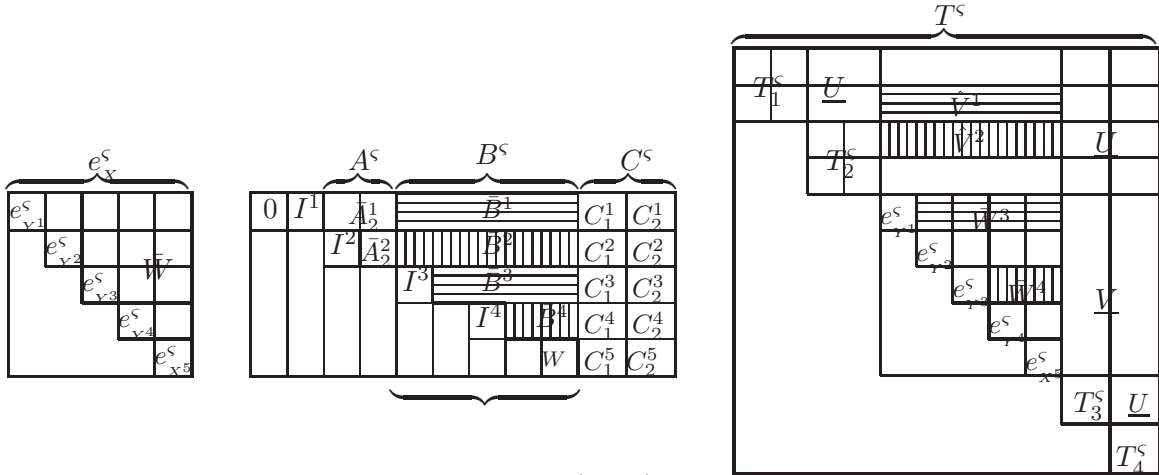
(vi) If $a_1^\zeta = c_{\tau \bar{p} \bar{q}}^\zeta$, since $\bar{b}_{\bar{p} q}^0 = \emptyset$ for all possible q by (iv), Formula (4.2-9) gives

$$\begin{aligned} \delta(c_{\tau \bar{p} \bar{q}}^\zeta) &= \sum_{i \leq h + \tau} \gamma_{\tau i}^0 \underline{v}_{i \bar{p} \bar{q}}^\zeta + \sum_{i, q} \bar{a}_{i \bar{p} q}^{\iota, 0} (\sum_l \zeta_{\tau i l} \underline{u}_{l q \bar{q}}) \\ &\quad + \sum_{i < \tau; q} \bar{c}_{i \bar{p} q}^{\iota, 0} (\sum_l \xi_{\tau i l} \underline{u}_{l q \bar{q}}) + \sum_{q < \bar{q}} \bar{c}_{\tau \bar{p} q}^{\iota, 0} w_{q \bar{q}} - \sum_{j \geq \iota, p > \bar{p}} w_{\bar{p} p} c_{\tau \bar{p} \bar{q}}^{j, 0}. \end{aligned}$$

If $\delta(c_{\tau \bar{p} \bar{q}}^\zeta) \neq 0$, then $\bar{\mathfrak{B}}^{\zeta+1}$ is given by $c_{\tau \bar{p} \bar{q}}^\zeta \mapsto \emptyset$, and a linear relation is added among $\underline{v}_{i \bar{p} \bar{q}}^\zeta$, $h < i \leq h + \tau$ and some $\underline{u}, w, \bar{w}$ -class elements, because $\underline{v}_{i \bar{p} \bar{q}}^\zeta$ for $i \leq h$ have already been replaced by some \underline{u} -class arrows given in (i).

The required $\underline{v}, \bar{u}, \bar{w}, \bar{v}$ -class dotted arrows are preserved, the pair $(\bar{\mathfrak{A}}^{\zeta+1}, \bar{\mathfrak{B}}^{\zeta+1})$ still satisfies assumption (i)-(vi). \square

The following picture shows a pseudo formal equation $\bar{\Theta}^\zeta$ of $(\bar{\mathfrak{A}}^\zeta, \bar{\mathfrak{B}}^\zeta)$ for $\kappa = 5, \gamma = 2$ in case (iii) ② of Theorem 4.3.4 only with effective arrows. From this, it is possible to see the correspondences of $(\bar{B}_\zeta^1, \hat{V}_\zeta^1)$, $(\bar{B}_\zeta^2, \hat{V}_\zeta^2)$, $(\bar{B}_\zeta^3, \bar{W}_\zeta^3)$, $(\bar{B}_\zeta^4, \bar{W}_\zeta^4)$ respectively.



Picture (4.5-2)

4.6 Regularizations on non-effective a class and all b class arrows

Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair, and the induced pair $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ be given by Theorem 4.3.4. Using the notation of Remark 3.4.6, we may assume that an induced local pair $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{A}}^s)$ of $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ is obtained by a sequence of reductions in the sense of Lemma 2.3.2. Set $a_1^s \mapsto (x)$, the induced pair $(\bar{\mathfrak{A}}^{s+1}, \bar{\mathfrak{A}}^{s+1})$ is obtained with $\bar{R} = k[x]$, and $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{A}}^t)$ in the case of MW5 is given by a series of regularizations. We will show in the last subsection, that x, a_1^t in $\bar{\mathfrak{B}}^t$ can be only split from some edges of $\bar{\mathfrak{B}}$.

It is clear by Lemma 4.5.1–4.5.3 that the non-effective a -class and all the b -class solid arrows are regularized during the reductions. Note that the conclusions of Lemmas 4.5.1–4.5.3 are still valid, if we deal with the linear relations over the fractional field $k(x)$ of the polynomial ring $k[x]$, or over the field $k(x, x_1)$ of two indeterminants instead of the base field k . Then the non-effective a -class and all the b -class solid arrows are regularized, which implies the following theorem.

Theorem 4.6.1 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair having at least two vertices, such that the induced local boc $\bar{\mathfrak{B}}_X$ is given by Formula (4.2-1), the pair is major, and the c -class arrows satisfy Formula (4.2-6). If $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ given by Theorem 4.3.4 has an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5 defined by Remark 3.4.6, then the parameter x and the first arrow a_1^t of $\bar{\mathfrak{B}}^t$ belong to \bar{a} or c -class.

Finally, let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair having at least two vertices, such that $\bar{\mathfrak{B}}_X$ is in case (i) of Classification 4.2.1. Then $\bar{\mathfrak{B}}$ has only a, b -class solid arrows, where b_1, \dots, b_n are all non-effective, and Formulae (4.2-5) is also suitable for a -class arrows: a_i for $i \in \Lambda$ satisfying the first formula of (4.2-5) are non-effective, while $\bar{a}_i = a_{h_i}$ for $1 \leq i \leq s$ satisfying the second one are effective. If there is an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5 according to Remark 3.4.6, there are following observations.

1) Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a pair with \mathcal{T} being trivial. The condition $(\text{BRC})'$ is constructed parallel to (BRC) in Condition 4.3.1 as follows.

- (i) Let $\mathcal{D} = \{d_1, \dots, d_q\}$ be a set of solid arrows, and $\mathcal{E} = \{e_1, \dots, e_p\}$ be another set of solid edges without any loop, such that $\mathcal{D} \cup \mathcal{E}$ forms the lowest non-zero row of the formal product Θ . And let $\mathcal{U} = \{u_1, \dots, u_q\}$ be a set of dotted arrows, while $\mathcal{W} = \mathcal{V} \setminus \mathcal{U}$.
- (ii) $\bar{\delta}(d_i)$ and $\bar{\delta}(e_i)$ satisfy the formulae of 4.3.1 (ii).

Then after a reduction given by Cases (i)–(iv) stated before Lemma 4.3.2, the induced pair still satisfies $(\text{BRC})'$. On the other hand, the original pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies $(\text{BRC})'$ parallel to Lemma 4.3.3. The proofs of the two facts are much easier than those of above two lemmas.

2) For constructing a reduction sequence from $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ up to $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$, what is needed is only the part (i) and (iii) ① of Theorem 4.3.4. In fact, the reduction block G^j is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for $j < \kappa$, and $G^\kappa = (1)$ or $(0 \ 1)$.

3) For further reductions, what is needed is only Theorem 4.5.3 (i)–(iii), then an induced pair $(\bar{\mathfrak{A}}^\varsigma, \bar{\mathfrak{B}}^\varsigma)$ is reached, where all the b -class, and non-effective a -class arrows are regularized step by step.

Corollary 4.6.2 Let $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ be a one-sided pair having at least two vertices, such that the induced local boc $\bar{\mathfrak{B}}_X$ is in case (i) of Classification 4.3.1. If $(\bar{\mathfrak{A}}^\kappa, \bar{\mathfrak{B}}^\kappa)$ given in 2) above has an induced pair $(\bar{\mathfrak{A}}^t, \bar{\mathfrak{B}}^t)$ in the case of MW5 defined by Remark 3.4.6, then the parameter x and the first arrow a_1^t of $\bar{\mathfrak{B}}^t$ belong to \bar{a} -class.

5. Non-homogeneity of bipartite matrix bimodule problems of wild type

This section is devoted to proving the non-homogeneous property for a wild bipartite matrix bimodule problem satisfying RDCC condition in the case of MW5. As a consequence, the main theorem 3 is proved in the last subsection.

5.1 An inspiring example

The purpose of this subsection is two folds: 1) classify the positions of the first arrows of bocses in the case of MW5 at formal products; 2) define a notion of the bordered matrix of a matrix, then prove a preliminary lemma on both matrices.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bimodule problem, which has a trivial vertex set $\mathcal{T} = \mathcal{T}_{(1)} \dot{\cup} \mathcal{T}_{(2)}$ and satisfies RBDCC condition, see Remark 1.4.4. Suppose $\mathfrak{A}' = (R', \mathcal{K}', \mathcal{M}', H')$ is an induced matrix bimodule problem in the case of MW5 defined by Remark 3.4.6. We show the classification of the position of the first arrow a_1' in the sum $H' + \Theta'$:

$$H' + \Theta' = H' + \sum_{i=1}^{n'} a_i' * A_i'. \quad (5.1-1)$$

Denote by (p, q') the leading position of A'_1 over \mathcal{T}' , which locates in the (\mathbf{p}, \mathbf{q}) -th leading block of some base matrix of \mathcal{M}_1 partitioned under \mathcal{T} . By the bipartite property and RDCC condition, \mathbf{q} is the index of a main block column, say $\mathbf{q} = \mathbf{q}_Z$ for some vertex $Z \in \mathcal{T}_{(2)}$.

Classification 5.1.1 Suppose the boc \mathfrak{B}' is in the case of MW5. There are two possible relations between the row index p and the row indices of the links of H' defined below Formula (2.3-7) in Formula (5.1-1):

case (I) $p <$ the row indices of all the links in the (\mathbf{p}, \mathbf{q}) -block of H' ;

case (II) $p \geq$ some row index of at least one link in the (\mathbf{p}, \mathbf{q}) -block of H' .

It is clear that there is no link above the (\mathbf{p}, \mathbf{q}) -th block, since \mathfrak{A}' is already local.

Lemma 5.1.2 Let p_x be the row index of x in H' , then $p_x > p$ in Classification 5.1.1.

Proof. Since x appears before the first arrow a'_1 in \mathfrak{B}' , $p_x \geq p$ by the order of reductions according to matrix indices. If $p_x = p$, then the parameter x locates at the left side of a'_1 in $H' + \Theta'$, $\delta(a'_1)$ contains only the terms of the form $\alpha xv, \alpha \in k$, which contradicts to the assumption that \mathfrak{B}' is in the case of MW5. Thus $p_x > p$. \square

Example 5.1.3 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair constructed by an algebra defined in Example 1.4.5. There is a reduction sequence $(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), (\mathfrak{A}^2, \mathfrak{B}^2), (\mathfrak{A}^3, \mathfrak{B}^3)$ given in Examples 2.4.5, such that the boc \mathfrak{B}^3 is strongly homogeneous in the case of MW5 described in Remark 3.1.7 (iii). In order to prove that $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous, another way different from the proof of MW1–MW4 must be found. More precisely, we will reconstruct a new reduction sequence based on the matrix \tilde{M} over $k[x]$ with the size vector $\tilde{m} = (2, 2, 2, 2, 2, 3, 3, 3, 3, 3)$:

$$\tilde{M} = \left(\begin{array}{c|c|c|c|c} 0 & 0 & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} & 0 \\ \hline & 0 & \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} & 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & x & 0 \end{smallmatrix} \\ \hline & & 0 & 0 & \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \\ \hline & & & 0 & \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \\ \hline & & & & 0 \end{array} \right).$$

There is a reduction sequence $(\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}}), (\tilde{\mathfrak{A}}^1, \tilde{\mathfrak{B}}^1), (\tilde{\mathfrak{A}}^2, \tilde{\mathfrak{B}}^2), (\tilde{\mathfrak{A}}^3, \tilde{\mathfrak{B}}^3)$ corresponding to the steps (i)–(iii) of Example 2.4.5, where the reduction from $\tilde{\mathfrak{B}}$ to $\tilde{\mathfrak{B}}^1$ is given by $a \mapsto (01)$ in the sense of Lemma 2.3.2. Thus b splits into b_1, b_2 in $\tilde{\mathfrak{B}}^1$, and set $b_1 \mapsto (0), b_2 \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ from $\tilde{\mathfrak{B}}^1$ to $\tilde{\mathfrak{B}}^2$. $\tilde{\mathfrak{B}}^3$ is obtained from $\tilde{\mathfrak{B}}^2$ by an edge reduction (0) , followed by a loop mutation, then four regularizations:

$$\tilde{H}^3 = \begin{pmatrix} 0 & 1_X & 0 \\ 0 & 0 & 1_X \end{pmatrix} * A + \begin{pmatrix} 0 & 0 & 1_X \\ 0 & 0 & 0 \end{pmatrix} * B + \begin{pmatrix} \emptyset & \emptyset & \emptyset \\ 0 & x1_X & \emptyset \end{pmatrix} * C.$$

The $(1, 5)$ -th block partitioned under \mathcal{T} in the formal equation of $(\tilde{\mathfrak{A}}^3, \tilde{\mathfrak{B}}^3)$ is of the form:

$$\begin{pmatrix} e & v \\ 0 & e \end{pmatrix} \begin{pmatrix} d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{pmatrix} + \begin{pmatrix} u_{11}^1 & u_{12}^1 \\ u_{21}^1 & u_{22}^1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} u_{11}^2 & u_{12}^2 \\ u_{21}^2 & u_{22}^2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{00}^2 & v_{01}^2 & v_{02}^2 \\ v_{10}^2 & v_{11}^2 & v_{12}^2 \\ 0 & vx & v_{11}^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{00}^1 & v_{01}^1 & v_{02}^1 \\ v_{10}^1 & v_{11}^1 & v_{12}^1 \\ v_{20}^1 & v_{21}^1 & v_{22}^1 \end{pmatrix} + \begin{pmatrix} d_{10} & d_{11} & d_{12} \\ d_{20} & d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} s_{00} & s_{01} & s_{02} \\ 0 & e & v \\ 0 & 0 & e \end{pmatrix}$$

with $e = e_X, s_{00} = e_Y$. The differentials of the solid arrows of $\tilde{\mathfrak{B}}^3$ can be read off as follows:

$$d_{20} : X \mapsto Y, \delta(d_{20}) = 0; \quad d_{21} : X \mapsto X, \delta(d_{21}) = xv - vx - d_{20}s_{01},$$

and $\bar{\delta}(d_{22}) = u_{21}^1 - v_{11}^2 - d_{20}s_{02} - d_{21}v$, $\bar{\delta}(d_{10}) = -v_{10}^2 - v_{20}^1 + vd_{20}$, $\bar{\delta}(d_{11}) = u_{11}^2 - v_{11}^2 - v_{21}^1 - d_{10}s_{01}$, $\bar{\delta}(d_{12}) = u_{11}^1 + u_{11}^2 - v_{12}^1 - v_{22}^1$, where $\bar{\delta}$ is obtained from δ by removing the monomials, which involve a solid arrow d_{22}, d_{10} or d_{11} . It is clear that the boc $\tilde{\mathfrak{B}}^3$ satisfies the hyperthesis of Proposition 3.4.5. In fact, as $d_{20} \mapsto (1)$, the solid loops $d_{21}, d_{22}, d_{10}, d_{11}, d_{12}$ will be regularized step by step, because $s_{01}, s_{02}, v_{10}^2, u_{11}^2, u_{11}^1$ are pairwise different dotted arrows. Therefore $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous.

Motivated by Example 5.1.3, the general cases are considered. Since the example satisfies Case (I) of Classification 5.1.1, we start from Case (I) in subsection 5.1–5.3.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bimodule problem satisfying RDCC condition. Let \mathfrak{A}' be an induced matrix bimodule problem with trivial R' , and let $\vartheta : R(\mathfrak{A}') \rightarrow R(\mathfrak{A})$ be the induced functor. Suppose $M = \vartheta(H'(k)) = \sum_j M_j * A_j \in R(\mathfrak{A})$ with a size vector $l \times \underline{n}$ over \mathcal{T} . Let $\mathbf{q} = \mathbf{q}_Z \in T_2$ for some $Z \in \mathcal{T}_2$. Define a size vector $l \times \underline{n}$ over \mathcal{T} , and construct a bordered matrix $\tilde{M} = \sum_j \tilde{M}_j * A_j \in R(\mathfrak{A})$ with 0 a zero column as follows:

$$\tilde{n}_j = \begin{cases} n_j, & \text{if } j \notin Z, \\ n_j + 1, & \text{if } j \in Z. \end{cases} \quad \tilde{M}_j = \begin{cases} M_j, & \text{if } A_j 1_Z = 0, \\ (0 \ M_j), & \text{if } A_j 1_Z = A_j. \end{cases} \quad (5.1-2)$$

Denote by $(p, q+1)$ the leading position of M , such that $q+1$ is the index of the first column of the \mathbf{q} -th block-column of M partitioned under \mathcal{T} . Denote by \tilde{q} the added column index of \tilde{M} , the column is sitting in the \mathbf{q}_Z -th block-column as the first one. Applying Theorem 2.4.1, the defining system \mathbb{E} of $\mathcal{K}'_0 \oplus \mathcal{K}'_1$ given by Formula (2.4-2), and a matrix equation $\tilde{\mathbb{E}}$ are considered:

$$\mathbb{E} : \Phi_l^1 M \equiv_{\prec(p, q+1)} M \Phi_{\underline{n}}^2, \quad \tilde{\mathbb{E}} : \Phi_l^1 \tilde{M} \equiv_{\prec(p, \tilde{q})} \tilde{M} \Phi_{\underline{n}}^2, \quad (5.1-3)$$

where the upper scripts 1, 2 on Φ stand for the left and right parts of the bipartite variable matrix Φ , the two sets of variables in two parts do not intersect. Since \mathfrak{A} satisfies RDCC condition, the main block-column in $\Phi_{\underline{n}}^2$ determined by $Z \in \mathcal{T}$ can be written as $\Phi_{\underline{n}, Z}^2 = (\Phi_1^2, \dots, \Phi_n^2)^T$, such that either $\Phi_l^2 = 0$ or $\Phi_l^2 = (z_{pq}^l) \neq 0$, where z_{pq}^l are variables over k .

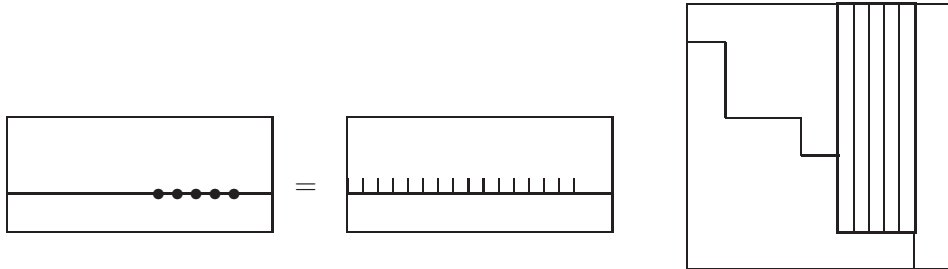
It is clear that the \tilde{q} -th column of $\Phi_l^1 \tilde{M}$ is a zero column, we may define two new matrix equations respectively:

$$\mathbb{E}_\tau : 0 \equiv_{\prec(p, q+1)} M \Phi_{\underline{n}}^2, \quad \tilde{\mathbb{E}}_\tau : 0 \equiv_{\prec(p, \tilde{q})} \tilde{M} \Phi_{\underline{n}}^2. \quad (5.1-4)$$

Taken any integer $p' \geq p$ and $1 \leq j \leq n_Z$, the $(p', q+j)$ -th entry of the right side of \mathbb{E}_τ is:

$$\sum_{\Phi_{\underline{n}, Z}^2 \neq 0} \sum_{q'} \alpha_{p'q'}^l z_{q', q+j}^l. \quad (5.1-5)$$

It is easy to see that $z_{q', q+j}^l$ for all possible j have the same coefficient $\alpha_{p', q'}^l$, the (p', q') -th entry of $H(k)$. In the picture below, $n_Z = 5, p' = p$, those five equations are indicated by five circles, and the five variables at the same row of $\Phi_{\underline{n}, Z}^2$ have the same coefficients.



Lemma 5.1.4 With the notations as above.

- (i) The $(p, q + j_1)$ -th equation is a linear combination of the previous equations in $\mathbb{I}\mathbb{E}_\tau$ if and only if so is the $(p, q + j_2)$ -th equation. Similarly, the same result is valid in $\tilde{\mathbb{I}\mathbb{E}}_\tau$.
- (ii) The equations in the system $\tilde{\mathbb{I}\mathbb{E}}$ (resp. $\tilde{\mathbb{I}\mathbb{E}}_\tau$) and those in $\mathbb{I}\mathbb{E}$ (resp. $\mathbb{I}\mathbb{E}_\tau$) are the same at the same positions of each main block column, whenever the added \tilde{q} -th column has been dropped from $\tilde{\mathbb{I}\mathbb{E}}$ (resp. $\tilde{\mathbb{I}\mathbb{E}}_\tau$).
- (iii) A subsystem of $\tilde{\mathbb{I}\mathbb{E}}$ (resp. $\tilde{\mathbb{I}\mathbb{E}}_\tau$) consisting of the \tilde{q} -column in both sides is $\tilde{\mathbb{I}\mathbb{E}}_{\tau\tilde{q}} : 0 \equiv \tilde{M}\Phi_{\underline{n},\tilde{q}}$, where $\Phi_{\underline{n},\tilde{q}}$ stands for the \tilde{q} -th column of $\Phi_{\underline{n}}$. And $\tilde{\mathbb{I}\mathbb{E}}_{\tau\tilde{q}}$ can be solved independently.
- (iv) If the $(p, q + j)$ -th equation is a linear combination of the previous equations of $\mathbb{I}\mathbb{E}$, then so is the p -th equation of $\tilde{\mathbb{I}\mathbb{E}}_{\tau\tilde{q}}$.

Proof (i) The assertion follows from Formula (5.1-5).

(ii) Recall that $\mathbf{q} \in Z$, denote by 0 the index of the first column (row) in the \mathbf{q}' -th block column (row) for any $\mathbf{q}' \in Z$. For any $X \in \mathcal{T}_{(1)}, Y \in \mathcal{T}_{(2)}$, set $1 \leq \alpha \leq n_X$ and $1 \leq \beta \leq n_Y$. We claim that the (α, β) -th equations in the $(\mathbf{h}, \mathbf{q}_Y)$ -th block of $\mathbb{I}\mathbb{E}$ and $\tilde{\mathbb{I}\mathbb{E}}$ for any $\mathbf{h} \in X$ are the same. In fact, the variable matrix Φ_l^1 in the two systems $\mathbb{I}\mathbb{E}$ and $\tilde{\mathbb{I}\mathbb{E}}$ is common; and the β -th column of the \mathbf{h} -th block row in M and \tilde{M} are the same in the left side of two equations. Now consider the right side of two equations. Let (M_1, \dots, M_t) be the α -row of the \mathbf{h} -th block row in M with $M_j = (\lambda_{j1}, \dots, \lambda_{jn_j})$ and that in \tilde{M} is $(\tilde{M}_1, \dots, \tilde{M}_t)$. Then $\tilde{M}_j = M_j, \forall j \notin Z$; but $\tilde{M}_j = (0, \lambda_{j1}, \dots, \lambda_{jn_j}), \forall j \in Z$. Let $(\Phi_1, \dots, \Phi_t)^T$ be the β -column of the \mathbf{q}_Y -th block column in $\Phi_{\underline{n}}^2$ with $\Phi_j = (x_{j1}, \dots, x_{jn_j})$ and that in $\tilde{\Phi}_{\underline{n}}^2$ is $(\tilde{\Phi}_1, \dots, \tilde{\Phi}_t)^T$, then $\tilde{\Phi}_j = \Phi_j, \forall j \notin Z$; but $\tilde{\Phi}_j = (x_{j0}, x_{j1}, \dots, x_{jn_j})^T, \forall j \in Z$. Thus the additional variables $x_{j0}, \forall j \in Z$, are killed by 0 in the right side of the equation of $\tilde{\mathbb{I}\mathbb{E}}$.

(iii) The \tilde{q} -column in the left side of the matrix equation $\tilde{\mathbb{I}\mathbb{E}}$ is a zero column. The variables of $\Phi_{\underline{n},\tilde{q}}$ are different from those in Φ_l and in the main block columns of $\Phi_{\underline{n}}$ except the \tilde{q} -th column.

(iv) If there exists some $1 \leq j \leq n_Z$, such that the $(p, q + j)$ -th equation is a linear combination of the previous equations in $\mathbb{I}\mathbb{E}$, then so is the $(p, q + j)$ -th equation in $\mathbb{I}\mathbb{E}_\tau$ after deleting Φ_l , since the variables of Φ_l and $\Phi_{\underline{n}}$ are different. Thus so is the $(p, \tilde{q} + j)$ -th equation in $\tilde{\mathbb{I}\mathbb{E}}_\tau$ by (ii), and so is the (p, \tilde{q}) -th equation in $\tilde{\mathbb{I}\mathbb{E}}_\tau$ by (i), finally, so is the p -th equation in $\tilde{\mathbb{I}\mathbb{E}}_{\tau\tilde{q}}$.

5.2 Bordered matrices in bipartite case

This subsection is devoted to constructing a reduction sequence based on a given sequence and a bordered matrix, which generalizes Example 5.1.3.

Let $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ be a bipartite matrix bimodule problem, which has a trivial \mathcal{T} and satisfies RDCC condition. Let $\mathfrak{A}^s = (R^s, \mathcal{K}^s, \mathcal{M}^s, H^s)$ be an induced matrix bimodule problem with R^s being local and trivial. Then there is a unique sequence of reductions in the sense of Lemma 2.3.2 by Corollary 2.3.5:

$$\mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^i, \mathfrak{A}^{i+1}, \dots, \mathfrak{A}^s. \quad (*)$$

Write $M = \vartheta^{0s}(H^s(k)) = \sum_j M_j * A_i \in R(\mathfrak{A})$ with a size vector $l \times \underline{n}$, where $M_j = G_s^{j+1}(k)$ are given by Formula (2.3-7). Suppose the size vector $l \times \tilde{\underline{n}}$ and the representation $\tilde{M} = \sum_j \tilde{M}_j * A_j \in R(\mathfrak{A})$ are defined by Formula (5.1-2).

Theorem 5.2.1 There exists a unique reduction sequence based on the sequence $(*)$:

$$\mathfrak{A}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^i, \tilde{\mathfrak{A}}^{i+1}, \dots, \tilde{\mathfrak{A}}^s \quad (\tilde{*})$$

where $\tilde{\mathfrak{A}}^i = (\tilde{R}^i, \tilde{\mathcal{K}}^i, \tilde{\mathcal{M}}^i, \tilde{H}^i)$, the reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ is a reduction or a composition of two reductions in the sense of Lemma 2.3.2. Moreover, $\tilde{\mathcal{T}}^s$ has two vertices, and

$$\tilde{\vartheta}^{0s}(\tilde{H}^s(k)) = \tilde{M}.$$

Proof We may assume that $l \times \underline{n}$ is sincere over \mathcal{T} . Otherwise, it is possible to obtain an induced problem \mathfrak{A}' by a suitable deletion, such that M has a sincere size vector over \mathfrak{A}' . In particular, \mathfrak{A}' is still bipartite and satisfies RDCC condition.

We will construct a sequence $(*)$ inductively. The original term in the sequence is $\tilde{\mathfrak{A}} = \mathfrak{A}$. Suppose that a sequence $\tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^i$ for some $0 \leq i < s$ has been constructed and $\vartheta^{0i} : R(\tilde{\mathfrak{A}}^i) \mapsto R(\tilde{\mathfrak{A}})$ is the induced functor, such that there exists a representation

$$\tilde{M}^i = \tilde{H}_{\tilde{m}^i}^i(k) + \sum_{j=1}^{n^i} \tilde{M}_j^i * \tilde{A}_j^i \in R(\tilde{\mathfrak{A}}^i), \quad \text{with} \quad \vartheta^{0i}(\tilde{M}^i) \simeq \tilde{M} \in R(\tilde{\mathfrak{A}}^0). \quad (5.2-1)$$

Write $M^i = \vartheta^{is}(H^s(k)) = H_{\underline{m}^i}^i(k) + \sum_{j=1}^{n^i} M_j^i * A_j^i \in R(\mathfrak{A}^i)$, where $M_1^i = G_s^{i+1}(k)$ by Formula (2.3-7), and is denoted by B for simplicity. The first column in the q -th main block-column of M under the partition \mathcal{T} is denoted by β . Now we are constructing \mathfrak{A}^{i+1} .

Case 1 $\tilde{\mathcal{T}}^i = \mathcal{T}^i$ and $\tilde{\mathfrak{A}}^i = \mathfrak{A}^i$.

1.1 $B \cap \beta$ is empty. Then $\tilde{G}^{i+1} = G^{i+1}$, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$.

Before giving the following cases, we claim that if $B \cap \beta$ is non-empty, B thus G^{i+1} can not be Weyr matrices. Otherwise, the first arrow a_1^i of \mathfrak{B}^i will be a loop. Since $\tilde{\mathcal{T}}^i = \mathcal{T}^i$, \tilde{a}_1^i will also be a loop and hence the numbers of rows and columns of \tilde{B} are the same. When the matrix B is enlarged by one column, then B is also enlarged by one row, which is a contradiction to the construction of \tilde{M} . So B is either (\emptyset) from a regularization or $\begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}$ from an edge reduction.

1.2 $B \cap \beta$ is non-empty, and $B = (\emptyset)$. Then $\tilde{B} = (\emptyset B)$ with \emptyset being a distinguished zero column, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$ by a regularization.

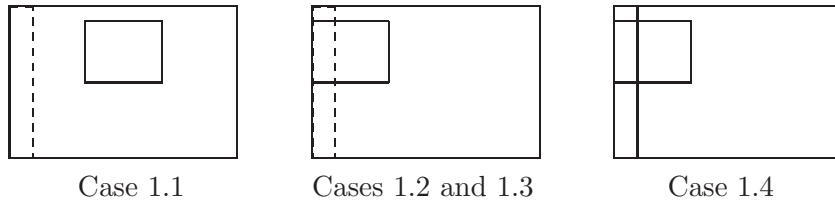
1.3 $B \cap \beta$ is non-empty, $B = \begin{pmatrix} 0 & I_r \\ 0 & 0 \end{pmatrix}$ and $r < \text{the number of columns of } B$. Then $\tilde{B} = (0 B)$ with 0 being a zero column, $\tilde{H}^{i+1} = H^{i+1}$ and $\tilde{\mathfrak{A}}^{i+1} = \mathfrak{A}^{i+1}$ by an edge reduction.

1.4 $B \cap \beta$ is non-empty, $B = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$. Then $\tilde{B} = (0 B)$ with 0 being a zero column. Recall from Formula (2.3-5) and Theorem 2.3.3, the following is defined:

$$\tilde{G}^{i+1} = \begin{cases} \begin{pmatrix} 0 & 1_{Z_2} \end{pmatrix}, & \text{if } G^{i+1} = (1_{Z_2}); \\ \begin{pmatrix} 0 & 1_{Z_2} \\ 0 & 0 \end{pmatrix}, & \text{if } G^{i+1} = \begin{pmatrix} 1_{Z_2} \\ 0 \end{pmatrix}. \end{cases}$$

Then $\tilde{H}^{i+1} = \sum_{X \in \mathcal{T}^i} I_X * H_X^i + \tilde{G}^{i+1} * A_1^i$. Consequently $\tilde{\mathfrak{A}}^{i+1}$ is induced from \mathfrak{A}^i by an edge reduction in the sense of Lemma 2.3.2.

We stress, that after the edge reduction in the subcase 1.4, $\tilde{\mathcal{T}}^{i+1} = \mathcal{T}^{i+1} \cup \{Y\}$, where Y is an equivalent class consisting of the indices of the added columns in the sum $\tilde{H}^{i+1} + \tilde{\Theta}^{i+1}$ of the pair $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1})$, and $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1}) \neq (\mathfrak{A}^{i+1}, \mathfrak{B}^{i+1})$ from this stage. The above B is show in four cases as a small block in the corresponding leading block partitioned under \mathcal{T} in \tilde{M} :



Case 2. $\tilde{\mathcal{T}}^i = \mathcal{T}^i \cup \{Y\}$.

2.1 $B \cap \beta$ is empty. Then $\tilde{B} = B$, $\tilde{G}^{i+1} = G^{i+1}$, and $\tilde{H}^{i+1} = \sum_{\tilde{X} \in \tilde{\mathcal{T}}^i} \tilde{I}_{\tilde{X}} * \tilde{H}_{\tilde{X}}^i + \tilde{G}^{i+1} * \tilde{A}_1$.

If $B \cap \beta$ is non-empty. Denote by \tilde{a}_0^i and \tilde{a}_1^i the first and the second solid arrows of $\tilde{\mathfrak{B}}^i$, which locate at (p^i, \tilde{q}_0^i) and $(p^i, \tilde{q}_0^i + 1)$ in the formal product $\tilde{\Theta}^i$ respectively.

2.2 $B \cap \beta$ is non-empty, and there exists some $1 \leq j \leq n_Z^i$, such that the $(p^i, q^i + j)$ -th equation is a linear combination of previous equations in \mathbb{E}_τ^i . Then $\delta(\tilde{a}_0^i) = 0$ by Lemma 5.1.4 (ii) then (i), and Corollary 2.4.2. Two reductions are made: the first one is an edge reduction by $\tilde{a}_0^i \mapsto (0)$; and the second one for \tilde{a}_1^i is made in the same way as that for a_1^i by Lemma 5.1.4 (ii). Then an induced problem $\tilde{\mathfrak{A}}^{i+1}$ is obtain, and $\tilde{B} = (0 \ B)$ with 0 being a zero column.

2.3 $B \cap \beta$ is non-empty, and for all $1 \leq j \leq m_Z^i$, the $(p^i, q^i + j)$ -th equation is not a linear combination of previous equations in \mathbb{E}_τ^i . Thus $\delta(\tilde{a}_0^i) \neq 0$ by Lemma 5.1.4 (i) and Corollary 2.4.2. And $\delta^0(\tilde{a}_j^i) \neq 0$ for any $1 \leq j \leq n_Z^i$ by 5.1.4 (i)–(ii) and Corollary 2.4.2. Then two regularizations $\tilde{a}_0^i \mapsto (\emptyset)$, $\tilde{a}_1^i \mapsto (\emptyset)$ are made, and $\tilde{B} = (\emptyset \ B)$ with \emptyset being a distinguished zero column.

In the cases 2.2 and 2.3, there are two reduction blocks $\tilde{G}^{i+1,0} = (0)$ or (\emptyset) , $\tilde{G}^{i+1,1} = G^{i+1}$, thus $\tilde{H}^{i+1} = \sum_{\tilde{X} \in \tilde{\tau}^i} \tilde{I}_{\tilde{X}} * \tilde{H}_{\tilde{X}}^i + \tilde{G}^{i+1,0} * \tilde{A}_0^i + \tilde{G}^{i+1,1} * \tilde{A}_1^i$.

By summing up all the cases, an induced pair $(\tilde{\mathfrak{A}}^{i+1}, \tilde{\mathfrak{B}}^{i+1})$ and a representation \tilde{M}^{i+1} with $\tilde{\vartheta}^{0,i+1}(\tilde{M}^{i+1}) \simeq \tilde{M}$ are obtained. The theorem follows by induction. \square

Corollary 5.2.2 The main diagonal block $\tilde{e}_Z^i, Z \in \mathcal{T}$, of $\tilde{\mathcal{K}}_0^i \oplus \tilde{\mathcal{K}}_1^i$ is of the form:

$$\begin{pmatrix} s_{00} & s_{01} & s_{02} & \cdots & s_{0m} \\ & s_{11} & s_{12} & \cdots & s_{1m} \\ & & s_{22} & \cdots & s_{2m} \\ & & & \ddots & \vdots \\ & & & & s_{mm} \end{pmatrix}.$$

where $m = n_Z^i$, $s_{01}, s_{02}, \dots, s_{0m}$ are dotted arrows of $\tilde{\mathfrak{B}}^i$.

Proof By the construction of \tilde{H}^i , the added “0-column” can be only 0 or \emptyset . Therefore, except s_{00} , the elements at the 0-th row: $s_{01}, s_{02}, \dots, s_{0m}$ do not appear in any equation of the defining system of $\tilde{\mathfrak{A}}^i$. Thus they are free. \square

5.3 Non-homogeneity in the case of MW5 and classification (I)

The discussion of this subsection is two folds: 1) extend the reduction sequence $(\tilde{*})$ of Theorem 5.2.1 into a sequence $(\tilde{*}')$, such that there is a parameter x appearing from the $(s+1)$ -th step; 2) prove that any bipartite pair with an induced minimal wild pair in the case of MW5 and Classification 5.1.1 (I) is not homogeneous.

Suppose we have a reduction sequence ending at \mathfrak{A}^t defined by Remark 3.4.6:

$$\mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^s, \mathfrak{A}^{s+1}, \dots, \mathfrak{A}^\epsilon, \dots, \mathfrak{A}^t = \mathfrak{A}', \quad (*')$$

where the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is in the sense of Lemma 2.3.2 for $1 \leq i < s$; \mathfrak{A}^s is local with $\delta(a_1^s) = 0$, set $a_1^s \mapsto (x)$, \mathfrak{A}^{s+1} has a parameter x locating at the (p_x, q_x) -position of H^{s+1} and $R^{s+1} = k[x]$; \mathfrak{A}^{i+1} is obtained from \mathfrak{A}^i by a regularization for $s < i < t$. The pair $(\mathfrak{A}^t, \mathfrak{B}^t)$ is in the case of MW5 given by Remark 3.4.6 and satisfying Classification 5.1.1 (I).

Note that the set of integers \mathcal{T}^i and its partition \mathcal{T}^i are all the same for $i = s, \dots, t$. Suppose the first arrow a_1^t of \mathfrak{B}^t locates at the (p, q') -th position in the formal product Θ^t with $q' = q + j$ for some $1 \leq j \leq n_Z^t$; the first arrow a_1^ϵ of \mathfrak{B}^ϵ locates at the $(p, q + 1)$ -th position in Θ^ϵ , where $q + 1$ is the index of the first column in the q -th block-column. The picture below shows the position of the first solid arrows in the formal products Θ^i of $(\mathfrak{A}^i, \mathfrak{B}^i)$ for $i = s, \epsilon, t$ (whenever the added \tilde{q}_0 -th column is ignored):

(ii) There exists some $f_\zeta(x) \neq 0$. Without loss of generality, it may be assumed that $f_\kappa(x) \neq 0$. Choose a new basis in the dual space $\text{Hom}_{k(x)}(\mathcal{K}_{\tau\bar{q}}^{(>h)} \otimes_{R_{\tau\bar{q}}^{(>h)}} k(x), k(x))$ at the first line of the formula below, the corresponding basis of $\mathcal{K}_{\tau\bar{q}}^{(>h)}$ is shown at the second line:

$$\begin{cases} u'_\zeta = u_\zeta, \\ U'_\zeta = U_\zeta - f_\zeta(x)/f_\kappa(x)U_\kappa; \end{cases} \quad \text{for } 1 \leq \zeta < \kappa; \quad \begin{cases} u'_\kappa = \sum_{\zeta=1}^{\kappa} f_\zeta(x)u_\zeta, \\ U'_\kappa = 1/f_\kappa(x)U_\kappa, \end{cases} \quad (5.3-4)$$

where $u'_\kappa = 0$ is the solution of the h -th equation in the system (5.3-3). Let $d^h(x) \in k[x]$ be the numerator of $f_\kappa(x)$, and $\gamma^h(x) = d^h(x)\gamma^{h+1}(x)$, then $R_{\tau\bar{q}}^{(>h-1)} = k[x, (\gamma^h(x))^{-1}]1_X \times k1_Y$. Thus $\mathcal{K}_{\tau\bar{q}}^{(>h-1)}$ has a quasi-basis $\{U'_\zeta \mid \zeta = 1, \dots, \kappa - 1\}$ over $R_{\tau\bar{q}}^{(>h-1)}$. The system $\tilde{\mathbb{F}}_{\tau\bar{q}}^{(>p-1)}$ with the solution space $\mathcal{K}_{\tau\bar{q}}^{(>p-1)}$ and polynomial $\gamma^p(x)$ is finally reached by inverse-order induction.

Suppose $R^i = k[x, \phi^i(x)^{-1}]$, and the row index of the first arrow of \mathfrak{B}^i in the formal product Θ^i is $p^i, p_x \leq p^i \leq p$ for $s \leq i \leq \epsilon$. Define

$$\tilde{\phi}^i(x) = \phi^i(x)\gamma^{p^i}(x) \in k[x], \quad \text{in particular} \quad \tilde{\phi}^t(x) = \phi^t(x)\gamma^p(x). \quad (5.3-5)$$

Now we deal with representations of \mathfrak{A}^i over the field $k(x)$ instead of over k . Suppose the matrix $M^i = \vartheta^{0i}(H^i(k[x, \tilde{\phi}^t(x)^{-1}])) = \sum_j M_j^i * A_j$ has a size vector $l \times \underline{n}$ over \mathcal{T} , and a matrix $\tilde{M}^i = \sum_j \tilde{M}_j^i * A_j$ of size vector $l \times \tilde{n}$ is defined by Formula (5.1-2). Returning to Theorem 2.4.1, the matrix equations for $i \geq s$ are defined as follows:

$$\begin{aligned} \mathbb{E}^i : \Phi_l M^i &\equiv_{\prec(p^i, q^i)} M^i \Phi_{\underline{n}}, & \tilde{\mathbb{E}}^i : \Phi_l \tilde{M}^i &\equiv_{\prec(p^i, q^i)} \tilde{M}^i \Phi_{\tilde{n}}; \\ \mathbb{E}_\tau^i : 0 &\equiv_{\prec(p^i, q^i)} M^i \Phi_{\underline{n}}, & \tilde{\mathbb{E}}_\tau^i : 0 &\equiv_{\prec(p^i, q^i)} \tilde{M}^i \Phi_{\tilde{n}}. \end{aligned} \quad (5.3-6)$$

Theorem 5.3.2 There exists a unique reduction sequence based on the sequence $(*)'$:

$$\mathfrak{A}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^s, \tilde{\mathfrak{A}}^{s+1}, \dots, \tilde{\mathfrak{A}}^\epsilon, \dots, \tilde{\mathfrak{A}}^t = \tilde{\mathfrak{A}}', \quad (\tilde{*}') \quad (5.3-7)$$

where the first part of the sequence up to $\tilde{\mathfrak{A}}^s$ is given by Theorem 5.2.1; the reduction from $\tilde{\mathfrak{A}}^s$ to $\tilde{\mathfrak{A}}^{s+1}$ is given by a loop mutation $a_1^{s+1} \mapsto (x)$, or an edge reduction (0) followed by a loop mutation (x) ; the reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ for $s < i < t$ is given by one regularization, or two regularizations, or a reduction given by Lemma 2.2.6, followed by a regularization.

Proof If the first arrow a_1^s of \mathfrak{B}^s does not locate at the $(q+1)$ -th column of the formal product Θ^s , a loop mutation from $\tilde{\mathfrak{A}}^s$ to $\tilde{\mathfrak{A}}^{s+1}$ is made. Otherwise an edge reduction is made by Remark 5.3.1, see 5.1.4 (iv) and Corollary 2.4.2 for details, then followed by a loop mutation.

Now suppose we have an induced bimodule problem $\tilde{\mathfrak{A}}^i$ for some $i > s$. If the first arrow a_1^i of \mathfrak{B}^i does not locate at the $(q+1)$ -th column of Θ^i , a regularization $\tilde{a}_1^i \mapsto \emptyset$ is made. Otherwise, there are two possibilities. ① There exists some $1 \leq j \leq n_Z$, the $(p^i, q+j)$ -th equation is a linear combination of the previous equations in \mathbb{E}_τ^i , then $\delta(\tilde{a}_0^i) = 0$ by Remark 5.3.1 and Corollary 2.4.2. Set $\tilde{a}_0^i \mapsto (0)$ by Lemma 2.6.6, $\tilde{a}_1^i \mapsto \emptyset$. ② Otherwise $\delta(\tilde{a}_0^i) \neq 0$. Set $\tilde{a}_0^i \mapsto \emptyset$ and $\tilde{a}_1^i \mapsto \emptyset$. The sequence $(\tilde{*}')$ is completed by induction as desired. \square

Corollary 5.3.3 If the boc \mathfrak{B}^t in the sequence $(*)'$ satisfies MW5 defined by Remark 3.4.6 and Classification 5.1.1 (I), then $\delta(\tilde{a}_0^\epsilon) = 0$ in $\tilde{\mathfrak{B}}^\epsilon$ in the sequence $(\tilde{*}')$.

Proof Since the first arrow a_1^t of \mathfrak{B}^t locates at the $(p, q+j)$ -th position with $\delta(a_1^t) = 0$, $\delta^0(a_j^\epsilon) = 0$ in $\tilde{\mathfrak{B}}^\epsilon$. Thus $\delta(\tilde{a}_0^\epsilon) = 0$ in $\tilde{\mathfrak{B}}^\epsilon$ by Remark 5.3.1 and Corollary 2.4.2. \square

Proposition 5.3.4 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with \mathcal{T} being trivial, such that $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ is a bipartite matrix bimodule problem satisfying RDCC condition. If there exists an induced

pair $(\mathfrak{A}', \mathfrak{B}')$ of $(\mathfrak{A}, \mathfrak{B})$ in the case of MW5 defined by Remark 3.4.6, and the sum $H' + \Theta'$ of $(\mathfrak{A}', \mathfrak{B}')$ satisfies Classification 5.1.1 (I), then \mathfrak{B} is not homogeneous.

Proof Suppose we have a sequence $(*)$ with $\mathfrak{B}' = \mathfrak{B}^t$, then there is a sequence $(\tilde{*})$ based on $(*)$ by Theorem 5.3.2. Corollary 5.3.3 tells that the first arrow \tilde{a}_0^ϵ of $\tilde{\mathfrak{B}}^\epsilon$ is an edge with $\delta(\tilde{a}_0^\epsilon) = 0$, and hence $\tilde{a}_0^\epsilon \mapsto (1)$ may be set according to Proposition 2.2.7. The induced pair is obviously local. Thus it is possible to use the triangular formulae of Subsection 3.3, and an induced pair $(\mathfrak{A}'', \mathfrak{B}'')$ is obtain in one of the cases (ii)-(iv) of Classification 3.3.5.

Case 1 If 3.3.5 (ii) is met, then $\tilde{\mathfrak{B}}^\epsilon$ satisfies the hypothesis of Proposition 3.4.5. It is done.

Case 2 If 3.3.5 (iii) is met, then $\tilde{\mathfrak{B}}''$ satisfies MW3, it is done by Proposition 3.4.3.

Case 3 If 3.3.5 (iv) is met, and \mathfrak{B}'' satisfies MW4, it is done by Proposition 3.4.4.

Case 4 If 3.3.5 (iv) is met, and \mathfrak{B}'' satisfies MW5, then there is an induced pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ in the case of MW5 defined by Remark 3.4.6. In this case the pair $(\mathfrak{A}^t, \mathfrak{B}^t)$ is denoted by $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$ in order to unify the notations. Suppose the first arrow \hat{a}_1^1 of $\hat{\mathfrak{B}}^1$ locates at the p^1 -th row in the formal product $\hat{\Theta}^1$. We claim that $p^1 < p$. In fact, the solid arrows \tilde{a}_j^ϵ for $j = 1, \dots, n_z$ at the p -th row of $\hat{\Theta}^\epsilon$ have differentials $\delta^0(\tilde{a}_j^\epsilon) = s_{0j} + \dots$ according to Corollary 5.2.2, and hence those arrows will be regularized step by step.

Repeating the above mentioned procedure for $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$, if one of the cases 1–3 is met, the procedure stops. Otherwise, if the case 4 is met repeatedly, there exist a sequence of local pairs and a decreasing sequence of the row indices:

$$\begin{array}{ccccccc} (\hat{\mathfrak{A}}, \hat{\mathfrak{B}}), & & (\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1), & & (\hat{\mathfrak{A}}^2, \hat{\mathfrak{B}}^2), & & \dots, & & (\hat{\mathfrak{A}}^\beta, \hat{\mathfrak{B}}^\beta), \\ p & > & p^1 & > & p^2 & > & \dots, & > & p^\beta. \end{array}$$

Since the number of the rows of \hat{H}^i for $i = 1, \dots, \beta$ is fixed, the procedure must stop at some stage β , where one of the cases 1–3 appears. The conclusion follows by induction. \square

5.4 Bordered matrices in one-sided case

In this subsection, a notion of reduced defining systems of Formula (2.4-3) is given for some induced pairs of a one-sided pair, which is different from Formula (4.1-7). Then some reduction sequences are constructed starting from one-sided pairs based on bordered matrices.

Let $(\mathfrak{A}, \mathfrak{B})$ be a bipartite pair satisfying RDCC condition, and $(\mathfrak{A}^r, \mathfrak{B}^r)$ be an induced pair with R^r being trivial. Suppose $(\mathfrak{A}^r, \mathfrak{B}^r)$ has a quotient-sub pair $((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$ denoted by $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ given in Formulae (4.1-1) and (4.1-2), where the vertex set $\bar{T} = \bar{T}_R \times \bar{T}_C \subseteq \mathcal{T}^r$, and $\bar{\mathfrak{B}}$ has a layer $L = (R; \omega; d_1, \dots, d_m; \bar{u}, \underline{u}, \bar{v}, \underline{v})$ by Definition 4.1.2.

Remark 5.4.1 Suppose $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$ with $\bar{\mathcal{T}}'$ being trivial is an induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$, then it is a quotient-sub-pair of $(\mathfrak{A}^{r+'}, \mathfrak{B}^{r+'})$ by Formula (4.1-5), where ι stands for some index. Recall Theorem 2.4.1 and the defining system $\mathbb{F}^{r, r+ \iota}$, the variable matrices $\Psi_{\underline{m}^{r, r+ \iota}}, \Psi_{\underline{m}^{r, r+ \iota}}^0$ given by Formula (2.4-3), we now define its reduced form consisting of the $(p^r, q^r), \dots, (p^r, q^r + m - 1)$ -th blocks of $\mathbb{F}^{r, r+ \iota}$ according to Remark 4.1.1. Suppose there is a functor $\bar{\vartheta}' : R(\bar{\mathfrak{A}}') \rightarrow R(\bar{\mathfrak{A}})$ acting on objects, F is defined below Formula (4.1-6), and $\bar{\vartheta}(F'(k))$ has a size vector $\underline{n} = (n_0; n_1, \dots, n_m)$ partitioned under $\bar{\mathcal{T}}$.

(i) Denote by \bar{Z}_0 the (p^r, p^r) -the square block of $\Psi_{\underline{m}^{r, r+ \iota}}$ of size n_0 with $p^r \in X^r$; by $\bar{Z}_{\xi\xi}$ the $(q^r + \xi - 1, q^r + \xi - 1)$ -th square block of size n_ξ with $q^r + \xi - 1 \in Y^r$ for any Y^r . Set

$$\bar{Z}_0 = Z_{X^r} = (z_{pq}^{X^r})_{n_0 \times n_0}, \quad \bar{Z}_{\xi\xi} = Z_{Y^r} = (z_{pq}^{Y^r})_{n_\xi \times n_\xi}.$$

(ii) Denote by \bar{Z}_ξ the $(p^r, q^r + \xi - 1)$ -block of $\Psi_{\underline{m}^{r, r+ \iota}}^0$ with size $n_0 \times n_\xi$, and by $\bar{Z}_{\eta\xi}$ the $(q^r + \eta - 1, q^r + \xi - 1)$ -block for $\eta < \xi$ of $\Psi_{\underline{m}^{r, r+ \iota}}$ with size $n_\eta \times n_\xi$. Write the matrices

$Z_j = (z_{pq}^j)_{n_{s(v_j^r)} \times n_{t(v_j^r)}}$ for all the dotted arrows of \mathfrak{B}^r , where $\{z_{pq}^j\}_{(p,q),j}$ are different variables over k . Suppose

$$\begin{aligned}\bar{Z}_\xi &= \sum_j \alpha_\xi^j Z_j, & \text{where } s(v_j^r) \ni p^r, t(v_j^r) \ni q^r + \xi - 1, & \alpha_\xi^j \in k; \\ \bar{Z}_{\eta\xi} &= \sum_j \beta_{\eta\xi}^j Z_j, & \text{where } s(v_j^r) \ni q^r + \eta - 1, t(v_j^r) \ni q^r + \xi - 1, & \beta_{\eta\xi}^j \in k.\end{aligned}$$

Return to the pair $(\bar{\mathfrak{A}}', \bar{\mathfrak{B}}')$, some indices \bar{p}, \bar{q}, \dots , in the formal product $\bar{\Theta}'$ will be used in order to distinguish with indices p, q, \dots , in the formal product $\Theta^{r+'}$ of $(\mathfrak{A}^{r+'}, \mathfrak{B}^{r+'})$ of Formula (5.1-1). Fix an integer $l \in \{1, \dots, m\}$ with $Y_l \neq X$ in Definition 4.1.2, thus $d_l : X \rightarrow Y_l$ is a solid edge. Suppose $(\bar{q}_l + 1)$ is the index of the first column in the l -th block-column of the formal product $\bar{\Theta}'$, such that $(\bar{p}, \bar{q}_l + 1)$ is the leading position of the first base matrix of $\bar{\mathcal{M}}'$. Write $\bar{M} = \bar{\vartheta}(F(k)) = (\bar{M}_1, \dots, \bar{M}_m) \in R(\bar{\mathfrak{A}})$ with the size vector \underline{n} over $\bar{\mathcal{T}}$. Then

$$\mathcal{F} : \bar{Z}_0 \bar{M} \equiv_{\prec(\bar{p}, \bar{q}_l + 1)} (\bar{Z}_1, \dots, \bar{Z}_m) + \bar{M}(\bar{Z}_{\eta\xi})_{1 \leq \eta \leq \xi \leq m} \quad (5.4-1)$$

is called a *reduced defining system* based on Theorem 2.4.1, which is different from $\bar{\mathcal{F}}$ given by Formula (4.1-7). Similarly as in Equation (5.1-4), there is an equation system:

$$\mathcal{F}_\tau : 0 \equiv_{\prec(\bar{p}, \bar{q}_l + 1)} (\bar{Z}_1, \dots, \bar{Z}_m) + \bar{M}(\bar{Z}_{\eta\xi})_{1 \leq \eta \leq \xi \leq m}. \quad (5.4-2)$$

Define a size vector $\tilde{n} = (\tilde{n}_0; \tilde{n}_1, \dots, \tilde{n}_m)$ over $\bar{\mathcal{T}}$ as follows: $\tilde{n}_j = n_j$ if $j \notin Y_l$; $\tilde{n}_j = n_j + 1$ if $j \in Y_l$. Construct a representation of $\bar{\mathfrak{A}}$ based on \bar{M} :

$$\tilde{M} = (\tilde{M}_1, \dots, \tilde{M}_m) \in R(\bar{\mathfrak{A}}), \quad \tilde{M}_j = \begin{cases} \bar{M}_j, & \text{if } j \notin Y_l; \\ (0 \bar{M}_j), & \text{if } j \in Y_l, \end{cases} \quad (5.4-3)$$

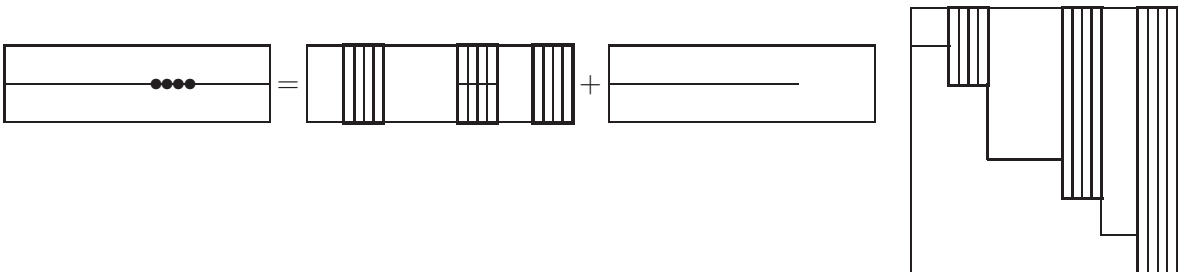
where 0 is a zero column. Write $\tilde{Z}_0, \tilde{Z}_{\eta\xi}$ the variable matrices of size $\tilde{n}_0 \times \tilde{n}_0, \tilde{n}_\xi \times \tilde{n}_\xi$; and $\tilde{Z}_\xi = \sum_j \alpha_\xi^j \tilde{Z}_j$ of size $\tilde{n}_0 \times \tilde{n}_\xi$, $\tilde{Z}_{\eta\xi} = \sum_j \beta_{\eta\xi}^j \tilde{Z}_j$ of size $\tilde{n}_\eta \times \tilde{n}_\xi$ respectively according to Remark 5.4.1. Denote by \tilde{q}_l the index of the first column of the l -th block-column of \tilde{M} , we obtain the following two matrix equations:

$$\begin{aligned}\tilde{\mathcal{F}} : \quad \tilde{Z}_0 \tilde{M} &\equiv_{\prec(\bar{p}, \tilde{q}_l)} (\tilde{Z}_1, \dots, \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{\eta\xi})_{1 \leq \eta \leq \xi \leq m} \\ \tilde{\mathcal{F}}_\tau : \quad 0 &\equiv_{\prec(\bar{p}, \tilde{q}_l)} (\tilde{Z}_1, \dots, \tilde{Z}_m) + \tilde{M}(\tilde{Z}_{\eta\xi})_{1 \leq \eta \leq \xi \leq m}.\end{aligned} \quad (5.4-4)$$

For any $l' \in Y_l$, denote by $\bar{q}_{l'} + 1$ the index of the first column of the l' -th block-column of \bar{M} . Set any integer $\bar{p}' \geq \bar{p}$ and $1 \leq h \leq n_{Y_l}$, the $(\bar{p}', \bar{q}_{l'} + h)$ -th entry in the right side of \mathcal{F}_τ of Formula (5.4-2) equals

$$\sum_{\bar{q}} \gamma_{\bar{p}'\bar{q}} z_{\bar{q}, \bar{q}_{l'}+h}^{Y_l} + \sum_{\bar{q}} \nu_{\bar{p}'\bar{q}} z_{\bar{q}, \bar{q}_{l'}+h}^j, \quad \gamma_{\bar{p}'\bar{q}}, \nu_{\bar{p}'\bar{q}} \in k, t(v_j) = Y_l. \quad (5.4-5)$$

The picture below shows four equations (abridged by four circles) of \mathcal{F}_τ . There are three solid edges ending at Y_l , i.e. $|Y_l| = 3$, and $n_{Y_l} = 4$. If l' is the second index of Y_l , then the equations at the $(\bar{p}', \bar{q}_{l'} + h)$ -th positions have the same coefficients for $h = 1, 2, 3, 4$.



Lemma 5.4.2 Being parallel to Lemma 5.1.4, there are following assertions.

(i) For any $1 \leq h_1, h_2 \leq n_l$ and any $l' \in Y$, the $(\bar{p}, \bar{q}_{l'} + h_1)$ -th equation is a linear combination of the previous equations in \mathcal{F}_τ , if and only if so is the $(\bar{p}, \bar{q}_{l'} + h_2)$ -th equation. Similarly, the same result is valid in $\tilde{\mathcal{F}}_\tau$.

(ii) The equations in the system $\tilde{\mathcal{F}}$ (resp. $\tilde{\mathcal{F}}_\tau$) and those in \mathcal{F} (resp. \mathcal{F}_τ) are the same at the same positions of each block column, whenever the added $\tilde{q}_{l'}$ -th columns for all $l' \in Y_l$ have been dropped from $\tilde{\mathcal{F}}$ (resp. $\tilde{\mathcal{F}}_\tau$).

(iii) A subsystem of $\tilde{\mathcal{F}}$ (resp. $\tilde{\mathcal{F}}_\tau$) consisting of the $\tilde{q}_{l'}$ -column in both sides is $\tilde{\mathcal{F}}_{\tau\tilde{q}_{l'}} : 0 \equiv \tilde{M}\Phi_{\tilde{n}, \tilde{q}_{l'}}$, where $\Phi_{\tilde{n}, \tilde{q}_{l'}}$ stands for the $\tilde{q}_{l'}$ -th column of $\Phi_{\tilde{n}}$. And the system $\{\tilde{\mathcal{F}}_{\tau\tilde{q}_{l'}} \mid \forall l' \in Y_l\}$ can be solved independently.

(iv) If the $(\bar{p}, \bar{q}_{l'} + h)$ -th equation for some $1 \leq h \leq n_y$ is a linear combination of the previous equations in \mathcal{F} , then so is the $(\bar{p}, \tilde{q}_{l'})$ -th equation in the system $\{\tilde{\mathcal{F}}_{\tau\tilde{q}_{l'}} \mid \forall l' \in Y_l\}$.

Proof (i)–(ii) See Proof (i)–(ii) of Lemma 5.1.4.

(iii) Note that $\forall l' \in Y_l$ the variables at the $\tilde{q}_{l'}$ -th column of $(\tilde{Z}_\xi)_{1 \leq \xi \leq m}$ and $(\tilde{Z}_{\eta\xi})_{1 \leq \eta \leq \xi \leq m}$ are different from those at the \bar{h} -th column for all $\bar{h} \neq \tilde{q}_{l'}, \forall l' \in Y$, and different from those in \tilde{Z}_0 .

(iv) See proof (iv) of 5.1.4.

Let $(\bar{\mathfrak{A}}^s, \bar{\mathfrak{B}}^s)$ be an induced pair of $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ with \bar{R}^s being trivial and local. Then there are two sequences of reduction in the sense of Lemma 2.3.2 according to Formula (4.1-5):

$$\begin{array}{cccccccc} \mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^{r-1}, & \mathfrak{A}^r, & \mathfrak{A}^{r+1}, & \dots, & \mathfrak{A}^{r+i}, & \mathfrak{A}^{r+i+1}, & \dots, & \mathfrak{A}^{r+s}; \\ \mathfrak{A}, & \mathfrak{A}^1, & \dots, & \mathfrak{A}^i, & \mathfrak{A}^{i+1}, & \dots, & \mathfrak{A}^s. \end{array} \quad (*)$$

Set $\bar{M} = \vartheta^{0s}(F^s(k))$ of size vector \underline{n} over $\bar{\mathcal{T}}$, a bordered matrix \tilde{M} of size vector \tilde{n} can be constructed according to Formula (5.4-3).

Theorem 5.4.3 Being parallel to Theorem 5.2.1, there exists a unique reduction sequence based on the sequence $(*)$, where $\tilde{\mathfrak{A}}$ stands for $\tilde{\mathfrak{A}}$ in order to simplify the notation:

$$\begin{array}{cccccccc} \tilde{\mathfrak{A}}, \tilde{\mathfrak{A}}^1, \dots, \tilde{\mathfrak{A}}^{r-1}, & \tilde{\mathfrak{A}}^r, & \tilde{\mathfrak{A}}^{r+1}, & \dots, & \tilde{\mathfrak{A}}^{r+i}, & \tilde{\mathfrak{A}}^{r+i+1}, & \dots, & \tilde{\mathfrak{A}}^{r+s}; \\ \tilde{\mathfrak{A}}, & \tilde{\mathfrak{A}}^1, & \dots, & \tilde{\mathfrak{A}}^i, & \tilde{\mathfrak{A}}^{i+1}, & \dots, & \tilde{\mathfrak{A}}^s. \end{array} \quad (\tilde{*})$$

(i) $\tilde{\mathfrak{A}}^i = \mathfrak{A}^i$ for $i = 0, 1, \dots, r$.

(ii) The reduction from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ is a reduction or a composition of two reductions in the sense of Lemma 2.3.2 for $i = 0, \dots, s-1$, such that $\tilde{\mathcal{T}}^s$ has two vertices, and $\tilde{\vartheta}^{0s}(\tilde{F}^s) = \tilde{M}$.

(iii) The reduction from $\tilde{\mathfrak{A}}^{r+i}$ to $\tilde{\mathfrak{A}}^{r+i+1}$ is done in the same way as that from \mathfrak{A}^i to \mathfrak{A}^{i+1} .

(iv) The diagonal block \tilde{e}_X in $\tilde{\mathcal{K}}^s$ of $\tilde{\mathfrak{A}}^s$ partitioned under $\tilde{\mathcal{T}}$ is of the form of Corollary 5.2.2.

Proof (i) is clear.

(ii) The proof is parallel to that of Theorem 5.2.1, the only difference appears in the item 1.4 of Case 1. Suppose an edge reduction is made from $\tilde{\mathfrak{A}}^i$ to $\tilde{\mathfrak{A}}^{i+1}$ with the reduction block G^{i+1} being at the l^0 -th block column with $l^0 \in Y_l$. Then \tilde{F}^{i+1} has a size vector $l^{i+1} \times \tilde{n}^{i+1}$ over $\tilde{\mathcal{T}}$ with $\tilde{n}_l^{i+1} = n_l^{i+1} + 1$, and a zero column is added into the l' -th block column from the left hand side for every $l' \in Y_l$.

(iii) follows from Formula (4.1-5).

(iv) The proof is parallel to that of Corollary 5.2.2. \square

Being parallel to $(*)'$ at the beginning of Subsection 5.3, there are following two sequences:

$$\begin{array}{cccccccc} \mathfrak{A}, \mathfrak{A}^1, \dots, \mathfrak{A}^{r-1}, & \mathfrak{A}^r, & \mathfrak{A}^{r+1}, & \dots, & \mathfrak{A}^{r+s}, & \mathfrak{A}^{r+s+1}, & \dots, & \mathfrak{A}^{r+\epsilon}, & \dots, & \mathfrak{A}^{r+t}, \\ \mathfrak{A}, & \mathfrak{A}^1, & \dots, & \mathfrak{A}^s, & \mathfrak{A}^{s+1}, & \dots, & \mathfrak{A}^\epsilon, & \dots, & \mathfrak{A}^t. \end{array} \quad (*')$$

The reductions from $\bar{\mathfrak{A}}$ to $\bar{\mathfrak{A}}^s$ is given by $(\bar{*})$; from $\bar{\mathfrak{A}}^s$ to $\bar{\mathfrak{A}}^{s+1}$ is a loop mutation and a parameter x appears; the reduction from $\bar{\mathfrak{A}}^i$ to $\bar{\mathfrak{A}}^{i+1}$ is a regularization for $i = s+1, \dots, t-1$. The first arrow of $\bar{\mathfrak{B}}^t$ locates at the $(\bar{p}, \bar{q} + j)$ -th position in the formal product $\bar{\Theta}^t$ for some $1 \leq j \leq n_l$, and that of $\bar{\mathfrak{B}}^\epsilon$ at $(\bar{p}, \bar{q} + 1)$ -th position in $\bar{\Theta}^\epsilon$. The pair $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ is minimal wild in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II).

Remark 5.4.4 (i) If the first arrow a_1^t of $\bar{\mathfrak{B}}^t$ is splitting from d_l of the one-sided boc $\bar{\mathfrak{B}}$, then $d_l : X \mapsto Y_l$ is an edge by Theorem 4.6.1 and Corollary 4.6.2. Therefore it is possible to apply Theorem 5.4.3 with respect to the vertex $Y_l \in \bar{\mathcal{T}}$ for the sequence $(\bar{*})$, and obtain the sequence $(\tilde{*})$.

(ii) We will describe how to determine \mathfrak{A}^r , thus $\bar{\mathfrak{A}}$, in the next subsection.

(iii) Being parallel to Formula (5.3-2), the equation system $\{\tilde{\mathcal{F}}_{\tau\bar{q}_{l'}} \mid \forall l' \in Y_l\}$ given by Lemma 5.4.2 (iii) is considered. Thus some polynomials $d^{j_{l'}}(x)$ are obtained for $l' \in Y_l, j = \bar{p}_x, \dots, \bar{p}$ inductively, by an analogous discussion as in the subsection 5.3. If $\bar{R}^t = k[x, \phi^t(x)^{-1}]$, define a polynomial similar to Formula (5.3-5):

$$\phi^t(x) = \phi^t(x) \prod_{j=\bar{p}_x}^{\bar{p}} \prod_{l' \in Y_l} d^{j_{l'}}(x).$$

Theorem 5.4.5 Being parallel to Theorem 5.3.2, there exist two unique reduction sequences based on the sequences $(\bar{*}')$:

$$\begin{array}{cccccccccccccccc} \bar{\mathfrak{A}}, \bar{\mathfrak{A}}^1, \dots, \bar{\mathfrak{A}}^{r-1} & \bar{\mathfrak{A}}^r, & \bar{\mathfrak{A}}^{r+1}, & \dots, & \bar{\mathfrak{A}}^{r+s}, & \bar{\mathfrak{A}}^{r+s+1}, & \dots, & \bar{\mathfrak{A}}^{r+\epsilon}, & \dots, & \bar{\mathfrak{A}}^{r+t}; \\ & \bar{\mathfrak{A}}, & \bar{\mathfrak{A}}^1, & \dots, & \bar{\mathfrak{A}}^s, & \bar{\mathfrak{A}}^{s+1}, & \dots, & \bar{\mathfrak{A}}^\epsilon, & \dots, & \bar{\mathfrak{A}}^t. \end{array} \quad (\tilde{*}')$$

(i) The first parts of the two sequences up to $\bar{\mathfrak{A}}^{r+s}$ and $\bar{\mathfrak{A}}^s$ respectively are given by $(\tilde{*})$.

(ii) $\bar{\mathfrak{A}}^{s+1}$ is induced from $\bar{\mathfrak{A}}^s$ by a loop mutation $a_1^{s+1} \mapsto (x)$, or an edge reduction (0) followed by a loop mutation (x) ; the reduction from $\bar{\mathfrak{A}}^i$ to $\bar{\mathfrak{A}}^{i+1}$ for $i > s$ is given by a regularization, or two regularizations, or a reduction given by Lemma 2.2.6 followed by a regularization.

(iii) The reduction from $\bar{\mathfrak{A}}^{r+i}$ to $\bar{\mathfrak{A}}^{r+i+1}$ is done in the same way as that from $\bar{\mathfrak{A}}^i$ to $\bar{\mathfrak{A}}^{i+1}$ for $s < i < t$.

(iv) If the boc $\bar{\mathfrak{B}}^t$ in the sequence $(\bar{*}')$ satisfies MW5 of Remark 3.4.6 and Classification 5.1.1 (II), then $\delta(\tilde{a}_0^\epsilon) = 0$ for the first arrow \tilde{a}_0^ϵ of $\bar{\mathfrak{B}}^\epsilon$ in $(\tilde{*}')$.

Proof (i) is obvious. The proof of (ii) is parallel to that of Theorem 5.3.2. (iii) follows from Formula (4.1-5). The proof of (iv) is parallel to that of Corollary 5.3.3 by Corollary 2.4.3. \square

5.5 Non-homogeneity in the case of MW5 and classification (II)

Suppose a bipartite pair $(\mathfrak{A}, \mathfrak{B})$ has an induced pair $(\mathfrak{A}', \mathfrak{B}')$ in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II) in this subsection. A one-sided quotient-sub pair is determined according to the position of the first arrow a_1' in the formal product $H' + \Theta'$; then the non-homogeneity of the pair $(\mathfrak{A}, \mathfrak{B})$ is proved.

Let \mathfrak{A} be a bipartite matrix bimodule problem satisfying RDCC condition. The sequence

$$(\mathfrak{A}, \mathfrak{B}), (\mathfrak{A}^1, \mathfrak{B}^1), \dots, (\mathfrak{A}^\varsigma, \mathfrak{A}^\varsigma), (\mathfrak{A}^{\varsigma+1}, \mathfrak{B}^{\varsigma+1}), \dots, (\mathfrak{A}^\tau, \mathfrak{B}^\tau) = (\mathfrak{A}', \mathfrak{B}') \quad (5.5-1)$$

satisfies the following conditions: R^i is trivial for $i \leq \varsigma$, the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is in the sense of Lemma 2.3.2 for $i < \varsigma$; \mathfrak{A}^ς is local with $\delta(a_1^\varsigma) = 0$, and $R^{\varsigma+1} = k[x]$ in $\mathfrak{B}^{\varsigma+1}$ after a loop mutation; finally, the reduction from \mathfrak{A}^i to \mathfrak{A}^{i+1} is a regularization for $i > \varsigma$, and $\mathfrak{B}^\tau = \mathfrak{B}'$ is in the case of MW5 of Remark 3.4.6 and Classification 5.1.1 (II). Suppose the index of the

first arrow a_1^τ of \mathfrak{B}^τ is (p^τ, q^τ) in the formal product Θ^τ , which is sitting at the (p, q) -th block partitioned under \mathcal{T} . According to Formula (2.3-7):

$$H^\tau = \sum_{i=1}^{\varsigma} G_\tau^i * A_1^{i-1} + (x) * A_1^\varsigma. \quad (5.5-2)$$

Following discussion will be focused on the reduction blocks G_τ^i of H^τ .

Let $i < \varsigma$, $\vartheta^{i\tau} : R(\mathfrak{A}^\tau) \rightarrow R(\mathfrak{A}^i)$ be the induced functor, and $\underline{n}^{i\tau} = \vartheta^{i\tau}(1, \dots, 1)$. There is a simple fact, that any row (column) index ρ of $H^i + \Theta^i$ in the pair $(\mathfrak{A}^i, \mathfrak{B}^i)$ determines a row (column) index $n_1^{i\tau} + \dots + n_\rho^{i\tau}$ of $H^\tau + \Theta^\tau$ in the pair $(\mathfrak{A}^\tau, \mathfrak{B}^\tau)$. Consequently, if the upper (resp. lower, left or right) boundaries of two reduction blocks $G_\tau^{j_1}, G_\tau^{j_2}$ in H^i are collinear, if and only if the corresponding boundaries of two splitting blocks $G_\tau^{j_1}, G_\tau^{j_2}$ in H^τ are collinear. The two blocks G_τ^j and $G_\tau^j(k)$ may not be distinguished for the sake of convenience in the following statements.

Remark 5.5.1 Consider the reduction blocks inside the (p, q) -th block partitioned under \mathcal{T} . The relative position of the upper boundaries of G_τ^i and G_τ^{i+1} in this block has three possibilities according to Formulae (2.3-3)–(2.3-5).

(i) The upper boundaries of G_τ^i and G_τ^{i+1} are collinear, and this occurs if and only if the reduction from \mathfrak{A}^{i-1} to \mathfrak{A}^i is given by one of the following reduction blocks: $G^i = \emptyset$ in a regularization; $G^i = (\lambda)$ in a loop reduction; $G^i = (0), (1)$ or $(0\ 1)$ in an edge reduction, moreover the right boundary of G_τ^i is not that of the (p, q) -th block. In this case their lower boundaries are also collinear.

(ii) The upper boundary of G_τ^{i+1} is strictly lower than that of G_τ^i , and this occurs if and only if $G^i = W$ of size being strictly bigger than 1 in a loop reduction, or $G^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in an edge reduction, and the right boundary of G_τ^i is not that of the (p, q) -th block. In this case, the lower boundaries of G_τ^i and G_τ^{i+1} are also collinear.

(iii) The lower boundary of G_τ^{i+1} is the upper boundary of G_τ^i , and this occurs if and only if the right boundary of G_τ^i coincides with that of the (p, q) -th block.

Collect all the reduction blocks of H^τ inside the (p, q) -th block, such that their upper boundaries are above or at that of a_1^τ :

$$G_\tau^{q_1}, G_\tau^{q_2}, \dots, G_\tau^{q_u}, \quad \text{with } 1 \leq q_1 < q_2 < \dots < q_u \leq \varsigma. \quad (5.5-3)$$

The set of reduction blocks $\{G_\tau^{q_i} \mid 1 \leq i \leq u\}$ in (5.5-3) is divided into h groups according to whether the upper boundaries of blocks are collinear or not, and denoted by ρ_j the common upper boundary of the blocks in the j -th group for $j = 1, \dots, h$, where ρ_{j+1} is strictly lower than ρ_j :

$$\{G_\tau^{q_{1,1}}, \dots, G_\tau^{q_{1,u_1}}\}, \dots, \{G_\tau^{q_{h,1}}, \dots, G_\tau^{q_{h,u_h}}\}, \quad u_1 + \dots + u_h = u. \quad (5.5-4)$$

The adjacent blocks $G_\tau^{q_{j,l}}$ and $G_\tau^{q_{j,l+1}}$ in the j -th group have two possibilities: ① $G_\tau^{q_{j,l}}$ is in case (i) of Remark 5.5.1, ② $G_\tau^{q_{j,l}}$ is in case (ii) of Remark 5.5.1. Then $G_\tau^{q_{j,l+1}}$ comes from the next reduction with $q_{j,l+1} = q_{j,l} + 1$ in ①. But in ②, $G_\tau^{q_{j,l+1}}$ follows by a sequence of reductions with the upper boundaries of the reduction blocks lower than that of a_1^τ , and the sequence includes at least one reduction in case (iii) of Remark 5.5.1. Finally, the sequence reaches $G_\tau^{q_{j,l+1}}$ with the upper boundary ρ_j as a neighbor of $G_\tau^{q_{j,l}}$. Thus $q_{j,l+1} > q_{j,l} + 1$.

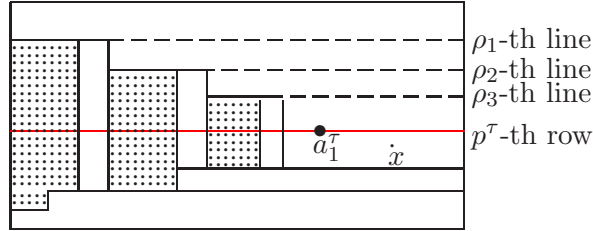
Lemma 5.5.2 The last block $G_\tau^{q_{j,u_j}}$ of the j -th group must be as in case (ii) of Remark 5.5.1 for $j = 1, \dots, h$.

Proof If $G_\tau^{q_{j,u_j}}$ is in case (iii) of 5.5.1, then ρ_j is lower than ρ_{j+1} for $j < h$, which is a contradiction to the grouping of Formula (5.5-4); and a_1^τ is sitting above ρ_h for $j = h$, which is a contradiction to the choice of the sequence (5.5-3).

Suppose $G_\tau^{q_j, u_j}$ is in case (i) of 5.5.1. Then for $j < h$, the upper boundaries of $G_\tau^{q_j, u_j}$ and $G_\tau^{q_j, u_j+1}$ coincide, which is a contradiction to the grouping of (5.5-4). For $j = h$, $G_\tau^{q_h, u_h} = 0, I, (0 I)$, or λI , or \emptyset with the height $d \geq 1$. Suppose the next reduction is still in the sense of Lemma 2.3.2 given by $G_\tau^{q_h, u_h+1}$, which is denoted by G'_τ for simplicity. If G'_τ is in the case (i) or (ii) of 5.5.1, then G'_τ and $G_\tau^{q_h, u_h}$ have the same upper boundary, a contradiction to the grouping of (5.5-4); if G'_τ is in case of 5.5.1 (iii), then a_1^τ locates above ρ_h , which is a contradiction to the choice of (5.5-3). Therefore the reduction in the sense of Lemma 2.3.2 should not be able to continue, and $r + s = q_{h, u_h}$ in the sequence $(\bar{*})'$ before Remark 5.4.4. Thus the height $d = 1$, the parameter x appears by a loop mutation. Since a_1^τ locates between the upper and lower boundaries of $G_\tau^{q_h, u_h}$, x and a_1^τ are sitting at the p^τ -th row simultaneously, a contradiction to Lemma 5.1.2. So $G_\tau^{q_j, u_j}$ is in the case of 5.5.1 (ii) as desired. \square

Definition 5.5.3 We define h rectangles in Θ^τ : for $j < h$, the j -th rectangle has the upper boundary ρ_j , the lower one ρ_{j+1} , and the left one is the right boundary of $G_\tau^{q_j, u_j}$, the right one is that of the (p, q) -th block. While the upper boundary of the h -th rectangle is ρ_h , lower boundary is that of $G_\tau^{q_h, u_h}$. The rectangle with upper boundary ρ_j is said to be the j -th *ladder*, there are altogether h ladders.

The picture below shows an example for $h = 3$. Three groups of Reduction blocks given in sawtooth patterns with some dots, but the last block in each group is given by a rectangle without dots. The upper boundaries of the three ladders are shown by dotted lines.



Lemma 5.5.4 Let index $r = q_{h, u_h} - 1$ in the sequence (5.5-1). We define a one-sided quotient-sub pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}}) = ((\mathfrak{A}^r)^{[m]}, (\mathfrak{B}^r)^{(m)})$ of the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$ consisting of the solid arrows d_1, \dots, d_m sitting at the p^τ -row as shown in Picture (4.1-1). Then

- (i) $m > 1$;
- (ii) all the reduction blocks in H^τ yielded from some split of d_2, \dots, d_m locate below the p^τ -th row;
- (iii) a_1^τ is split from d_l with $l > 1$. If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies the hypothesis of Theorem 4.6.1 or Corollary 4.6.2, then d_l is a solid edge.
- (iv) $\varsigma = r + s$ and $\tau = r + t$. Therefore the sequence (5.5-1) coincides with the first sequence of $(\bar{*})'$ given before Remark 5.4.4.

Proof (i) follows from Lemma 5.5.2.

(ii) comes from the choice of the reduction blocks of Formula (5.5-3).

(iii) and (iv) are obvious. \square

Remark 5.5.5 (i) $H^r + \Theta^r$ of the pair $(\mathfrak{A}^r, \mathfrak{B}^r)$ has also h ladders in the (p, q) -th block. The boundaries of the j -th ladder of $H^r + \Theta^r$ is derived from that of the j -th ladder of $H^r + \Theta^r$ for $j = 1, \dots, h$ according to the simple fact stated before Remark 5.5.1.

(ii) Let $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ be the induced local pair at X of $(\mathfrak{A}^r, \mathfrak{B}^r)$, denote by h_X the number of the inheriting ladders of $H_X^r + \Theta_X^r$ from $H^r + \Theta^r$, then $h_X \leq h$.

(iii) Return to sequence $(\bar{*})'$ and $(\tilde{*})'$ in Subsection 5.4. It is easy to see that $\tilde{H}^{r+t} + \tilde{\Theta}^{r+t}$ has also h ladders, and the number of rows in the h -th ladder in $\tilde{H}^{r+t} + \tilde{\Theta}^{r+t}$ is the same as that in $H^{r+t} + \Theta^{r+t}$.

Proposition 5.5.6 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with \mathcal{T} trivial, such that $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ is a bipartite matrix bimodule problem satisfying RDCC condition. If there exists an induced pair $(\mathfrak{A}', \mathfrak{B}')$ of $(\mathfrak{A}, \mathfrak{B})$ in the case of MW5 defined by Remark 3.4.6, and the sum $H' + \Theta'$ of $(\mathfrak{A}', \mathfrak{B}')$ satisfies Classification 5.1.1 (II), then \mathfrak{B} is not homogeneous.

Proof Suppose the induced pair $(\mathfrak{A}', \mathfrak{B}')$ is the last term $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ in the first sequence of Formula $(\tilde{*})$ given before Remark 5.4.4. Keep the notations in the two sequences of $(\tilde{*})$.

We assume in addition that the number of the ladders in $H^{r+t} + \Theta^{r+t}$ is minimal with respect to the property of Classification 5.1.1 (II).

(I) Let X be given by Definition 4.1.2. If the local pair $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ is wild, using the triangular Formulae of Subsection 3.3, an induced minimal wild local pair $((\mathfrak{A}_X^r)', (\mathfrak{B}_X^r)')$ with the parameter x' and the first arrow a'_1 is obtained.

(I-1) If $(\mathfrak{B}_X^r)'$ is in the case of MW3, or MW4, or MW5 with $H'_X + \Theta'_X$ being in the case of Classification 5.1.1 (I), then $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous by Proposition 3.4.3, or 3.4.4, or 5.3.4, it is done.

(I-2) If $(\mathfrak{B}_X^r)'$ is in the case of MW5 and Classification 5.1.1 (II), then the number of the inheriting ladders h_X in $H_X^r + \Theta_X^r$ does not exceed h by Remark 5.5.5 (ii). Suppose a'_1 locates at the h' -ladder. If $h' < h_X \leq h$, or $h' = h_X < h$, then it contradicts to the minimality assumption on the number of ladders. If $h' = h_X = h$, since this ladder contains only one row by Lemma 5.5.4, x' and a'_1 must locate at the same row, a contradiction to Lemma 5.1.2.

(II) Suppose $(\mathfrak{A}_X^r, \mathfrak{B}_X^r)$ is tame infinite, and its quotient-sub-pair $(\bar{\mathfrak{A}}_X, \bar{\mathfrak{B}}_X)$ is in case (ii) of Classification 4.2.1.

(II-1) If the one-sided pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies the hypothesis of Lemma 4.2.3 or 4.4.1, then $(\mathfrak{A}, \mathfrak{B})$ is not homogeneous. In fact the loop \bar{b} is the unique effective loop of both $\bar{\mathfrak{B}}_X$ and $\bar{\mathfrak{B}}_X^r$.

(II-2) If $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ satisfies the hypothesis of Theorem 4.4.2, then the triangular formulae given in Subsection 3.3 can be used for the local wild pair $(\mathfrak{A}^{r+2l}, \mathfrak{B}^{r+2l})$ given in Proof 4) of 4.4.2. If the cases of MW3, MW4, or MW5 and Classification 5.1.1 (I) are reached, it is done. If MW5 and Classification 5.1.1 (II) is met again, the first arrow must be outside of the h -th ladder, which is a contradiction to the minimal number assumption of the ladders.

(III) Now the following two cases are considered. First, \mathfrak{B}_X^r is tame infinite, $\bar{\mathfrak{B}}_X$ is in the case (ii) of Classification 4.2.1, the pair $(\bar{\mathfrak{A}}, \bar{\mathfrak{B}})$ is major and the c -class arrows satisfy Formula (4.2-6). Second, \mathfrak{B}_X^r is tame infinite or finite, and $\bar{\mathfrak{B}}_X$ is finite.

Then in both cases d_l of $\bar{\mathfrak{B}}$, from which a'_1 is split, is a solid edge by Lemma 5.5.4 (iii). Consequently, Formula $(\tilde{*})$ of Theorem 5.4.3 can be used with respect to d_l . Keep the notations in the two sequences of $(\tilde{*})$ of Theorem 5.4.5.

Since $\delta(\tilde{a}_0^\epsilon) = 0$ in the pair $(\tilde{\mathfrak{A}}^\epsilon, \tilde{\mathfrak{B}}^\epsilon)$ of $(\tilde{*})$ by 5.4.5 (iv), set the edge $\tilde{a}_0^\epsilon \mapsto (1)$ by Proposition 2.2.7. Then all the other arrows splitting from d_l at the same row are regularized in the further reductions by 5.4.3 (iv). The induced pair is obviously local and tame infinite or wild type. Then it is possible to use the triangular formulae of Subsection 3.3 once again, and an induced pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ in the cases (ii)-(iv) of Classification 3.3.5 is obtained.

(III-1) If the induced local pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is tame infinite, then the two-point pair $(\tilde{\mathfrak{A}}^{r+\epsilon}, \tilde{\mathfrak{B}}^{r+\epsilon})$ satisfies the hypothesis of Proposition 3.4.5, it is done.

(III-2) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW3, or MW4, or MW5 of Remark 3.4.6 and Classification 5.1.1 (I), it is done.

(III-3) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW5 of Remark 3.4.6 and classification 5.1.1 (II), and suppose in addition, the first arrow of $\hat{\mathfrak{B}}^1$ locates at the h_1 -th ladder with $h_1 < h$, then there is a contradiction to the minimality number assumption of the ladders.

(III-4) If $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ is in the case of MW5 of Remark 3.4.6 and classification 5.1.1 (II), and suppose in addition, the first arrow of $\hat{\mathfrak{B}}^1$ locates still at the h -th ladder, it is needed to do induction on some pairs of integers. Denote by σ the number of the rows in the h -th ladder of

$H^{r+t} + \Theta^{r+t}$, which is a constant after making some bordered matrices by Remark 5.5.5 (iii). And denote by m the number of the solid arrows in the pair $(\mathfrak{A}, \mathfrak{B})$ in Formula (4.1-1), which is also a constant. Define a finite set with σm pairs:

$$\mathcal{S} = \{(\varrho, \zeta) \mid 1 \leq \varrho \leq \sigma, \zeta = 1, \dots, m\},$$

ordered by $(\varrho^1, \zeta^1) \prec (\varrho^2, \zeta^2) \iff \varrho^1 > \varrho^2, \text{ or } \varrho^1 = \varrho^2, \zeta^1 < \zeta^2.$

In order to unify notations, the induced minimal wild local pair $(\mathfrak{A}^{r+t}, \mathfrak{B}^{r+t})$ in $(\bar{*})$ is denoted by $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$. Let $(\varrho, \zeta) = (\bar{p}, l) \in \mathcal{S}$, where \bar{p} is the row-index of the first arrow $\hat{a}_1 = a_1^t$ in $\bar{\Theta}^t$; l is the subscript of the edge d_l , from which a_1^t is split, since $F^t + \bar{\Theta}^t$ is contained in the h -th ladder by Lemma 5.5.4. Similarly, let $(\varrho^1, \zeta^1) \in \mathcal{S}$ be determined by the first arrow \hat{a}_1^1 of $\hat{\mathfrak{B}}^1$. Theorem 5.4.3 (iv) ensures that $(\varrho, \zeta) \prec (\varrho^1, \zeta^1)$.

Now the procedure (III) is started once again from the pair $(\hat{\mathfrak{A}}^1, \hat{\mathfrak{B}}^1)$ instead of $(\hat{\mathfrak{A}}, \hat{\mathfrak{B}})$. If (III-4) appears repeatedly, then after a finite number of steps, an induced pair of (III-1)–(III-3) is reached by induction on \mathcal{S} . \square

5.6 Proof of the main theorem

It is ready to prove Theorem 3 given in the introduction.

Theorem 5.6.1 Let $(\mathfrak{A}, \mathfrak{B})$ be a pair with \mathcal{T} trivial, such that $\mathfrak{A} = (R, \mathcal{K}, \mathcal{M}, H = 0)$ is a bipartite matrix bimodule problem satisfying RDCC condition. If \mathfrak{B} is of wild type, then \mathfrak{B} is not homogeneous.

Proof There exists an induced boc \mathfrak{B}' of the wild boc \mathfrak{B} , which is in one of the cases of MW1–MW5 according to Classification 3.3.2. Proposition 3.4.1–3.4.4 proved that if \mathfrak{B}' is in the case of MW1–MW4, then \mathfrak{B} is not homogeneous. When \mathfrak{A} is bipartite and satisfies RDCC condition, Proposition 5.3.4 and 5.5.6 proved that if the induced boc \mathfrak{B}' is in the case of MW5 of Remark 3.4.6, then \mathfrak{B} is not homogeneous. \square

Proof of Theorem 3 Let Λ be a finite-dimensional basic algebra over an algebraically closed field k . We claim that if Λ is of wild representation type, then $\text{mod-}\Lambda$ is not homogeneous.

In fact, let \mathfrak{A} be the matrix bimodule problem associated with Λ . Then \mathfrak{A} is bipartite and satisfies RDCC condition by Remark 1.4.4, and it is representation wild type. Therefore the associated boc \mathfrak{B} is not homogeneous by Theorem 5.6.1. Note that there is a one-to-one correspondence between the set of equivalent classes of almost split sequences in $\text{mod-}\Lambda$ and that of almost split conflations in $R(\mathfrak{B})$ except finitely many equivalent classes of such sequences, see [B2] and [ZZ]. Therefore $\text{mod-}\Lambda$ is not homogeneous. \square

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